Learning and Adaptation as a Source of Market Failure

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Abstract

A dynamic model of financial markets with learning and strategy adoption is demonstrated to produce episodes of market failure. Traders engage in learning to improve their understanding of the relationship between observed prices and future payoffs. Traders also choose between strategies based on fundamental research or on extracting information from market data. The two evolutionary processes interact and can produce extreme price deviations and conditions for which no market clearing price exists.

JEL codes: G14, C62, D82

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1. Introduction

Markets are populated by an extraordinary number of traders employing a wide variety of strategies. Many attempt to extract rent through trading. Casual observation of the market suggests a lack of agreement among market participant as to the true price determination process. The disagreement extends to issues of market efficiency and how deviations from efficiency can best be exploited. Trading strategies range from those traders who engage in the process of analyzing individual companies, industries, and aggregate economic information to make forecasts of future profits in an attempt to find undervalued securities. Others traders employ the structure of markets in an effort to find and exploit price inconsistencies. Still others employ market-based information as a tool to extract information regarding future price innovations. Of this latter approach, there is near infinite variety in the tools employed and just as many trader offered explanations for how particular signals should be interpreted and why they might work.

The behavior of price in the market reflects the beliefs and trading strategies employed. This feeds back to shape the beliefs of the traders and affects the traders’ choice between strategies. The model employed in this analysis captures this interaction between traders and the market price.

A financial market model based on a heterogeneous population of traders is examined. Traders choose one of two strategies capable of generating profits in the model. Traders may engage in fundamental research, which provides each adherent with information concerning the future payments. Alternatively, traders may elect to employ market-based information. The price does contain information, reflecting the demand of the informed fundamental traders, that can be employed to generate profit. The traders, though, are required to learn the model for extracting information from the price.
The learning process is one of two governing dynamic processes in the model. The other is the selection of whether to rely on fundamental information or market-based information in devising a trade for the current period. The population proportion evolves over time as traders respond to past performance.

The model offers rich dynamics that provide a narrative of how extreme market prices can arise from the efforts of individual traders to learn and adapt to changing market conditions.

2. Model

2.1 An overview

The model employed follows from Goldbaum (2006). The market consists of two assets, a risk-free bond and a single infinitely lived dividend paying asset. The traders have two strategies for accessing information about the future dividend. They may engage in individual fundamental research or rely on market-based information such as the price.

The dynamic financial market model employed offers a number of parallels to the static model of Grossman and Stiglitz (1980). The market includes two groups of traders; a group of informed traders accessing fundamental information and a group of uninformed trader who have chosen to forgo direct access to the fundamental information in favor of extracting that information from the price (market data). As is developed, if the uninformed traders have the correct model for extracting information, then the uninformed strategy offers the higher payoff, thus providing an incentive for traders to free-ride on the research effort of other traders. The model thus incorporates the paradox of the Grossman and Stiglitz (GS) model that the ability to free-ride requires that someone else engages in the fundamental research without compensation.

There are a number of important features distinguishing the developed model from that of GS and the dynamic extensions of GS such as Bray (1982) and Routledge (1996). Structural
differences in how the market is established, apart from the dynamics, play key roles in the behavior of the model in its dynamic setting.

The risky security is infinitely lived and thus the market realization in one period has an impact on the returns offered by the market in the preceding and following periods. In the tradition repeated GS model, including Bray (1982) and Routledge (1996), the termination of the asset at the end of the period makes each period’s market independent.

Another deviation from GS is in the nature of the private information learned by the informed fundamental traders. In GS, the same private signal is shared by all of the informed traders. To implement the GS type of information for an infinitely lived asset, one would suppose a dividend process $d_{t+1} = \theta_{t+1} + \varepsilon_{t+1}$, providing knowledge of $\theta_{t+1}$ to the all of the informed traders. With $\varepsilon_{t+1}$ unobserved, each informed trader is assured of possessing all of the information that exists in the market. Thus, in GS, the uninformed traders trade at an informational disadvantage but find compensation in the lower cost of their information.

In the current setting, a fundamental trader engages in individual research from which is learned next period’s dividend with idiosyncratic error, equation (4) developed below. The fundamental trader’s uncertainty is in the error of his or her own signal. The result is a Radner (1979) type market that, when efficient, filters out the idiosyncratic errors of the individual fundamental traders so that the resulting market price reveals the underlying information.

There is both an aesthetic and pragmatic advantage to this approach relative to GS. Aesthetically, there is no single ultimate information source and no ultimately informed investor or investor group. Information among the fundamental traders is heterogeneous. The market-based analysis offers a true advantage if the information contained in the price is less noisy than individual effort of fundamental analysis. This explanation for market based trading seems
preferable to one based on an arbitrary cost advantage. Certainly, some of the most sophisticated approaches to market-based trading are as effort and cost intensive as is fundamental analysis.

Pragmatically, the nature of the private information also changes the nature of the market solution. In both the GS and Goldbaum (2006) models, a fixed mapping between the private information and the price at the rational expectations equilibrium allows the market-based uninformed investor to back the private information out of the observed price. In GS, the rational expectations equilibrium price is a simple function of the private information, 

$$p_t^{REE} = f(\theta_{t+1}).$$

Since the extraction is simple it is reasonable to presume traders can employ deduction to find the REE.

In Goldbaum (2006), the relationship between price and the underlying payoff is state dependent. Let \(n_t\) represent the proportion of the market that is fundamentally informed, as will be demonstrated, 

$$p_t^{REE} = f(\theta_{t+1}, n_t).$$

Traders have to account for the decisions of the other traders to be able to correctly interpret the information content of the price. The market is thus a more challenging environment for the rational trader to exploit through pure deduction, undermining the existence of a REE while offering legitimacy to populating markets with heterogeneous traders employing a variety of strategies and learning from observation, as will be demonstrated.

2.2 Fundamentals and market equilibrium

A large but finite number of agents, indexed by \(i = 1, ..., N\), trade a risky asset and a risk-free bond. The risk-free bond, with a price of one, pays \(R\). The risky asset is purchased at the market determined price, \(p_t\), in period \(t\). In \(t+1\), it pays a stochastic dividend \(d_{t+1}\), and sells for the market determined price \(p_{t+1}\). The market participants are aware that the stochastic dividend follows a commonly known AR(1) process centered around \(d_0\):
\[ d_i = d_0 + \eta_i , \]

with \( \eta_i = \phi \eta_{t-1} + \epsilon_i, \epsilon_i \sim IIDN(0, \sigma^2_\epsilon) \). \hfill (1)

Let \( z_i = p_i + d_i \) and \( \theta_\mu = 1/\gamma \sigma^2_\mu \) with \( \sigma^2_\mu = \text{var}_\mu(z_{t+1}) \) indicating the conditional variance. The parameter \( \gamma \) is the coefficient of absolute risk aversion. In each period, each myopic trader maximizes a negative exponential utility function on one period ahead wealth conditional on his individual information set (to be developed below). This produces the demand for the risky asset,

\[ q_\mu(p_i) = \frac{E_\mu(p_{t+1} + d_{t+1} - Rp_i)}{\gamma \text{Var}_\mu(p_{t+1} + d_{t+1})} = \theta_\mu (E_\mu(z_{t+1}) - Rp_i) . \hfill (2) \]

Assume \( K \) strategies, indexed by \( k \in \{1,2,\ldots,K\} \), for estimating \( z_{t+1} \). In a Walrasian equilibrium, the market price equates supply and demand for the asset. Supply is fixed to avoid the exogenous introduction of noise. For convenience, set fixed net supply of the risky asset to zero. Let \( N_k \) be the total number of traders employing information \( I^k \). Let \( q^k_i \) be the per capita demand for the risky security among group \( k \) traders, \( q^k_i(p_i) = \frac{1}{N_k} \sum_{i=1}^{N_k} q_\mu(p_i) \). Let \( n^k_i \), \( 0 \leq n^k_i \leq 1 \), be the proportion of the trader population employing strategy \( k \), \( \sum_k n^k_i = 1 \). The price \( p_i \) clears the market by solving

\[ 0 = \sum_{k=1}^{K} n^k_i q^k_i(p_i) . \hfill (3) \]

2.3 Information

2.3.1 The Fundamental Trader

The estimate of the future payoff is in the nature of Hellwig (1980). Trader \( i \)'s private research conducted at time \( t \) produces a noisy signal of the intrinsic value at time \( t+1 \). This
information gathering process is captured by the time $t$ private signal, $s_t$, centered around $d_{t+1}$, but subject to an idiosyncratic error term,

$$s_t = d_{t+1} + e_t = d_0 + n_{t+1} + e_t,$$  \hspace{1cm} (4)

with $e_t \sim IIDN(0, \sigma^2_{e})$.

A linear projection of $\eta_{t+1}$ onto the information set produces the fundamental investor's mean squared error minimizing forecast

$$E_{t}(\eta_{t+1} \mid \eta_t, s_t) = (1-\beta)\phi e_t + \beta s_t,$$  \hspace{1cm} (5)

where the weight $\beta$ is known based on the traders' knowledge of the dividend and information processes,

$$\beta = \frac{\text{cov}(\eta_{t+1}, s_t)}{\text{var}(s_t)} = \frac{\sigma^2_{e}}{\sigma^2_{e} + \sigma^2_{v}}.$$

The "fundamental" price prevails in a market populated exclusively by fundamental investors. Derive the fundamental price by using the estimate (5) in (2),

$$p_t^F = p_t^F(\eta_t, \eta_{t+1}) = b_0 + b_1^F \eta_t + b_2^F \eta_{t+1} + v_t,$$  \hspace{1cm} (6)

with $b_0 = d_0 / (R - 1)$, $b_1^F = (1-\beta)\phi / (R - \phi)$, $b_2^F = \beta / (R - \phi)$ and $v_t = \frac{\beta}{R - \phi} \frac{1}{N} \sum_i e_i$. For large $N$, the impact of the idiosyncratic signal noise on the price is negligible. Assume a sufficiently large $N$ such that the $v_t$ term can be dropped.\footnote{Formally, $v_t$ is $o(1)$.}

Reflecting a mixture of current public information ($\eta_t$) and private information ($\eta_{t+1}$), $p_t^F$ is generally not efficient. The exceptions are when $\sigma^2_{e} \rightarrow \infty$, $p_t^F \rightarrow p_t^0 = \frac{d_0}{R - 1} + \frac{\phi}{R - \phi} \eta_t$, and
for $\sigma^2 = 0$, $p^F_t = p^EM_t = \frac{d_0}{R-1} + \frac{1}{R-}\eta_{t+1}$. As the private signal becomes increasingly noisy, the price converges to reflect just the public information contained in $d_t$, and the price is Semistrong-form efficient according to the Fama (1970) definitions of efficiency. When traders receive a perfect noise-free signal on the next period’s dividend, the price fully reflects the $d_{t+1}$ based value producing a Strong-form efficient price.²

Fundamental traders rely on (6) in forming demand. Plug (6) back into (2) to derive the average demand of the group of fundamental traders,

$$q^F_t (p_t) = \theta^F_t \left( \frac{Rd_0}{R-1} + \frac{R}{R-}\phi((1-\beta)\phi_t + \beta\eta_{t+1}) - Rp_t \right).$$

(7)

2.3.2 Market-based traders

The market-based traders model the relationship between the payoff and current market observables. They consistently estimate $z_t$ using the information $p_t$ and $\eta_t$,

$$E_M(z_{t+1}) = c_0 + c_1 p_t + c_2 \eta_t.$$  

(8)

Let $x'_t = [1, p_t, \eta_t]$ and $c = [c_0, c_1, c_2]$. The market-based traders all rely on the same public information, and thus all employ the same forecast, $E(z_{t+1} \mid x_t) = cx_t$. Per capita demand among market-based traders is thus

$$q^M_t = \theta^M_t (cx_t - Rp_t).$$

(9)

2.4 Price Formation

With $K = 2$, let $n^F_t = n^F$, and thus $1-n^M_t = n^M_t$. From (3),

$$0 = n^F_t q^F_t + (1-n^F_t)q^M_t$$

(10)

Use (7), (9), and (10) to solve for the market clearing price. A consistent price function takes the

² Efficient market is in the nature of Radner (1979). The price reflects the aggregation of the fundamental traders knowledge since no individual trader knows $d_{t+1}$ with certainty.
form

\[ p_t = p_t(n_t, e_t) = b_0 + b_1(n_t, e_t)n_t + b_2(n_t, e_t)n_{t+1} \quad (11) \]

with

\[ b_0 = d_0 / (R - 1), \quad (12) \]

\[ b_1(n_t, e_t) = \Psi(n_t, e_t)^{-1}(n_t^\theta)^{\phi} - \frac{n^\theta}{\phi}(1 - \beta)\phi + (1 - n_t)c_2^n \theta^M, \]

\[ b_2(n_t, e_t) = \Psi(n_t, e_t)^{-1}(n_t^\theta)^{\phi} - \frac{n^\theta}{\phi}\beta, \]

\[ \Psi(n, e_t) = n_R^\theta + (1 - n_t)(R - c_n^M)\theta^M, \]

\[ \sigma^2_{\varepsilon_f} = ((1 - \beta)^{\phi} + b_2(n_t, e_t)^2)\sigma^2_{\varepsilon}, \text{ and} \]

\[ \sigma^2_{\varepsilon_m} = ((\frac{n}{R - \phi} - c_n^n b_2(n_t, e_t)^2 + b_2(n_t, e_t)^2)\sigma^2_{\varepsilon}. \quad (13) \]

2.5 A rational expectations equilibrium

Consider a fixed \( n_t = n \forall t \) where \( 0 < n \leq 1 \). There is an \( n \) dependent rational expectations equilibrium (REE) given by:

\[ b_1^*(n) = \frac{n^\theta (1 - \beta)^{\phi}}{(R - \phi)(n^\theta + (1 - n)\theta^M)}, \quad b_2^*(n) = \frac{n^\theta \beta + (1 - n)(n^\theta + (1 - n)\theta^M)}, \quad (14) \]

\[ c_1^*(n) = \frac{R}{b_2^*(n)} = \frac{R(n^\theta (1 - \beta)^{\phi}}{n^\theta \beta + (1 - n)\theta^M}, \quad (15) \]

\[ c_2^*(n) = \frac{\phi}{R - \phi}(R - c_1^*(n)) = \frac{n(1 - \beta)^{\phi}}{(R - \phi)(n^\theta \beta + (1 - n)\theta^M)}, \]

\[ \sigma^2_{\varepsilon_f}(n^2) = \frac{(1 - \beta)(R/(R - \phi))^2 + b_2^*(n)^2)\sigma^2_{\varepsilon}, \text{ and} \]

\[ \sigma^2_{\varepsilon_m}(n^2) = b_2^*(n)^2\sigma^2_{\varepsilon}. \quad (16) \]

At this equilibrium, the market based traders employ the correct model for predicting the future payoff given the available public information, \( p_t \) and \( d_t \). At the fixed point, the market
traders correctly extract $d_{i+1}$ from the from $p_i$. Importantly, the equilibrium model employed by the market-based traders and the equilibrium price depend on the value of $n$. The value of $n$ determines how much the equilibrium price reflects the current publicly know $d_i$ and how much it reflects the future $d_{i+1}$ embedded in the private information. In other words, from (14) it is clear that $p^*_i$ exists between $p^0_i$ and $p^{EM}_i$. As $n \to 0$, the weight shifts towards $d_{i+1}$.

The positive value of $c_{ts}$, the regression coefficient on price, reflects the market-based traders’ positively sloped demand function. The market-based traders interpret an increase in the price as an indication that the informed traders have received good news concerning the future dividends. Since the demand by the fundamental traders alone produces a price that only partially reflects the future dividend, there is room for the market-based traders to earn a profit from the information they can extract from the price. The REE requires that the market-based traders account for their own impact on the price, which depends on $n$.

Knowing $n$ is equivalent to knowing the investment strategy employed by the other market participants. If this information is available to the traders, the REE can be deduced from the available public information. If the information is not available, then the REE cannot be deduced through rational analysis of the market structure.

Note the GS like discontinuity in the REE solution at $n = 0$: for $n \to 0$ $c^*_1(n) \to R$, $c^*_2(n) \to 0$, $b^*_1(n) \to 0$, and $b^*_2(n) \to 1/(R-\phi)$ so that $p^*_i \to p^{EM}_i$ but at $n = 0$, $c^*_1(0) = R$, $c^*_2(0) = 0$ is a consistent solution producing $b^*_1(0) = \phi/(R-\phi)$ and $b^*_2(0) = 0$ so that $p^*_i = p^0_i$. There is no value of $n$ at which $p^*_i(n) = p^{EM}_i$.

### 2.6 Learning

Suppose the traders are unable to derive the REE solution. This could be the result of
bounded rationality, or it may simply be the result of removing \( n \) from the trader’s information set. The GS REE model for extracting information from the price of is independent of \( n \) for \( n_t > 0 \), enabling rational traders to find the REE solution without knowledge of \( n \). From (14) – (16), it is clear that this model requires knowledge of \( n \) to derive the correct relationship between \( p_t \) and payoff or, in other words, the correct coefficients of (8).

Unable to derive the REE solution analytically, it is reasonable to think of traders deriving the relationship between observables and payoff from observation. Suppose traders use market generated data to estimate the coefficients of (8). Traders’ evolving beliefs are captured by the standard recursive updating algorithm:

\[
\begin{align*}
e_t &= e_{t-1} + \lambda_t (Q_{t-1}^{-1}x_{t-2}(z_{t-1} - e_{t-1}x_{t-2}))', \\
Q_t &= Q_{t-1} + \lambda_t (x_{t-1}x_{t-1}' - Q_{t-1}),
\end{align*}
\]

given \((e_0, Q_0)\). Parameter \( \lambda_t \) is the gain parameter. If \( \lambda_t = 1/t \), then the algorithm becomes the least-squares learning model developed by Marcet and Sargent (1989a, 1989b). In this case, with \( n \) fixed at some value \( 0 < n \leq 1 \), the learning process has the REE as the fixed point to the evolutionary beliefs.

Thus, through observation, taking a statistical approach allows the traders to learn the REE, thereby overcoming the hurdle of incomplete knowledge of the model they populate. The degree of sophistication necessary to behave rationally is greatly reduced by the statistical approach. The market-based traders are able to reach the REE freed from the requirement that they know the trading strategies of the other market participants, as captured by \( n \). Further, statistically derived model of information extraction correctly accounts for the market traders’ own impact on the price without cognizant effort.

To simplify the discussion and analysis that follows, it is useful to reduce the market-based
traders model to a single parameter. Knowledge of the underlying dividend process and of coefficient $c_0$’s independence from $n$ allows the traders to correctly deduce the value of $c_0$.

Further, presume that $c_{2t}$ is consistent with $c_{lt}$ according to (15). The analysis by Goldbaum (2006) finds four critical values to $c_{lt}$. In addition to $c^*_1(n)$ in (15), are $c^-_1 = R$, $c^*_1 = c^+_1(n)$, and $\overline{c}_1 = \overline{c}_1(n)$ with $c^-_1 < c^*_1(n) < c^+_1(n) < \overline{c}_1(n)$, listed in Table 1.

Table 1: Critical values for $c_{lt}$

<table>
<thead>
<tr>
<th>Beliefs of the Market Traders</th>
<th>Market Clearing Price (based on positive innovation in $d_{t+1}$)</th>
<th>Defining equation</th>
<th>Relevant Features</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{lt} &lt; c^-_1$</td>
<td>$p_t &lt; p^F_t$</td>
<td>---</td>
<td>$F$ traders outperform $M$ traders in expectation</td>
</tr>
<tr>
<td>$c_{lt} = c^-_1$</td>
<td>$p_t = p^F_t$</td>
<td>$q^M_t = 0$</td>
<td>$c_{2t} = 0, b_{lt} = b^F_t, b_{2t} = b^E_{2t}$, $E(profits) = 0$</td>
</tr>
<tr>
<td>$c^-<em>1 &lt; c</em>{lt} &lt; c^*_1$</td>
<td>$p^F_t &lt; p_t &lt; p^*_t$</td>
<td>---</td>
<td>$M$ traders outperform $F$ traders in expectation</td>
</tr>
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<td>---</td>
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</tr>
<tr>
<td>$c^*<em>1 &lt; c</em>{lt} &lt; c^+_{lt}$</td>
<td>$p^*_t &lt; p_t &lt; p^{EM}_t$</td>
<td>---</td>
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</tr>
<tr>
<td>$c_{lt} = c^+_{lt}$</td>
<td>$p_t = p^{EM}_t$</td>
<td>$b_{lt} = 0$</td>
<td>$E(profits) = 0$</td>
</tr>
<tr>
<td>$c^*<em>1 &lt; c</em>{lt} &lt; \overline{c}_1$</td>
<td>$p^{EM}_t &lt; p_t &lt; \infty$</td>
<td>---</td>
<td>$F$ traders outperform $M$ traders in expectation</td>
</tr>
<tr>
<td>$c_{lt} \rightarrow \overline{c}_1$</td>
<td>$p_t \rightarrow \infty$</td>
<td>$\Psi_t \rightarrow 0$</td>
<td>$E(F$ trader profits) $\rightarrow \infty$</td>
</tr>
<tr>
<td>$\overline{c}<em>1 &lt; c</em>{lt}$</td>
<td>$p_t &lt; 0$</td>
<td>---</td>
<td>A price satisfying (11) exists but is of debatable economic importance</td>
</tr>
</tbody>
</table>

At $c_{lt} = c^-_1$, the market-based traders, believing that the asset grows at the risk-free rate, have no reason to trade and thus $p_t = p^F_t$. For all values of $c_{lt} \neq c^-_1$, the market traders exert pressure on the price. As $c_{lt}$ increases, the upward sloping demand curve of the market-based traders becomes increasingly steep. At $c_{lt} = c^*_1$, the market-based model correctly extracting $d_{t+1}$ from the price. For $c_{lt} > c^*_1$, the market-based traders overestimate the innovation in $d_{t+1}$. 

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The exuberance of their response pushes price innovations beyond the value of $p_t^*$. In the case of $c_{tt} = c_t^+$, the demand of the market-based traders pushes the price fully to $p_t^{EM}$.

As $c_{tt} \rightarrow \overline{c}_t$, the market-based traders become increasingly responsive to the smallest signal from the price, driving the market clearing price to $\pm\infty$, as determined by the direction of the dividend innovation. For $c_{tt} > \overline{c}_t$, the upward sloping demand of the market-based traders dominates the downward sloped demand of the fundamental traders. In this case the market demand function is upwards sloping.

Table 1 characterizes the impact of $c_{tt}$ on the market for a positive innovation in $d_{t+1}$, so that $p_t^f < p_t^* < p_t^{EM} < \infty$.

Figure 1 includes $c_t^-$ and illustrates the dependence of $c_t^-(n)$, $c_t^+(n)$, and $\overline{c}_t(n)$ on $n$. The exact function for $c_t^-(n)$, and $\overline{c}_t(n)$ depends on how the traders estimate $\sigma_t^2$, a topic that is discussed below.

Consider deviations in $p_t$ from $p_t^*$. Let $dp_t = |p_t - p_t^*|$. Further, let $dp_t^+ = |p_t - p_t^*|$ if $c_{tt} > c_t^+$ and let $dp_t^- = |p_t - p_t^*|$ if $c_{tt} < c_t^+$. The two scenarios have to be considered separately because the impact of the bias in $c_{tt}$ is not symmetric. For any given magnitude in the price deviation, there are two possible candidates of $c_{tt}$ that can produce it. In both cases, the larger the bias, the larger the price deviation.

**Proposition 1**: Assume a fixed $n$.
1. If $\lambda_t = 1/t$, then $\Pr(dp_t > \delta) \longrightarrow 0$ for $\delta > 0$.
2. If $\lambda_t = \lambda_t, 0 < \lambda_t(n) < 1$ then $\Pr(dp_t > \delta) \longrightarrow k$ for $\delta > 0$ and $0 < k < 1$.

Further, $k$ is increasing in $\lambda$ and in $n$

**Proof of (1.1)**: Borrowing notation from Marcet and Sargent (1989), let $S(c)$ be an operator that maps the parameter $c$ onto itself. In the case of the learning process, $S(c)$ maps $c$ into the
projection coefficients. From (15), \( c_i = \frac{R}{R - \phi b_i} \) so that, according to (12),
\[
S(c_i) = (nR\theta^F + (1 - n)(R - c_i)\theta^H)/(n\beta\theta^F)
\]
and
\[
\frac{\partial S(c) - c}{\partial c}
\]
\( \left|_{c_n = R} \right. = -1 - \frac{1}{n} \frac{1}{\theta^H} \frac{\partial \theta}{\partial \theta} < 0 \), conforming to the local stability criteria developed in Marcet and Sargent (1989a).

Proof of (1.2): To evaluate the convergence properties under constant gain, consider a random variable \( x \) governed by the process, \( x_t = x_0 + \varepsilon_t \) with \( \varepsilon \sim IIDN(0, \sigma^2) \). Let \( \theta_t \) be a sample statistic based on the realizations of \( x_t \), \( \theta_t = \theta_{t-1} + \lambda(x_t - \theta_{t-1}) \). For \( \lambda = 0 \), \( \theta_t \) is a constant. For \( \lambda \in (0,1] \), by the Central Limit Theorem, as \( t \to \infty \), \( \theta_t \xrightarrow{d} IIDN(x_0, \sigma_0^2) \) where \( \sigma_0^2 = \frac{\lambda}{2 - \lambda} \sigma^2 \).

The challenge is that \( c_{c_t} \) need not be asymptotically normal and may not have a finite second moment. This situation arises when \( \lambda \) is high (so that memory is short) and \( n \) is low. In this case \( \Pr(c_{c_t} \geq \bar{c}_t) \) is non-negligible asymptotically. Simulations confirm that \((\lambda, n) \) combinations that generate a sufficiently large distance, \( \bar{c}_t - c^*_t \), such that the \( \Pr(c_{c_t} \geq \bar{c}_t) \) is negligible asymptotically, then \( c_{c_t} \) converges to an effectively stable distribution with a variance that is increasing in \( \lambda \).

Given a distribution for \( c_{c_t} \) at time \( t \), the explanation for why the likelihood of a large price deviation increases as \( n \to 0 \) can be seen in Figure 1. The convergence of \( c^*_t, c^+_t, c^+_t, \) and \( \bar{c}_t \) towards a point at \( R \) reflects a decrease in the market’s ability to tolerate error in the beliefs of the traders. The fewer fundamental traders there are to stabilize the price, the greater the impact of an error in the market-based traders’ model. Constant gain ensures a persistent amount of error in the estimate of \( c_{c_t} \). The lower is \( n \), the greater the price error this estimation error generates.

The functions \( c^*_t, c^+_t, \) and \( \bar{c}_t \) are all invertible. Thus, given a particular \( c_{c_t} > R \), using \( c^*_t \), there is a unique value of \( n \) for which the particular \( c_{c_t} \) is the correct model, call this \( n^*(c_{c_t}) \).

Using \( c^+_t \), there is a unique value of \( n \) for which the particular \( c_{c_t} > R \) produces equal profits, call this \( n^*(c_{c_t}) \). Finally, using \( \bar{c}_t \), there is a unique value of \( n \) for which the particular
$c_{it}$ produces infinite prices, call this $n(c_{it})$. For a given $c_{it}$, $n(c_{it}) < n^*(c_{it}) < n^*(c_{it})$. For the trader unaware of $n$ no values of $c_{it} > R$ can be excluded as unreasonable. The unbounded beliefs of the market-based traders should thus be considered a source of market instability.

### 3.2 Accounting for forecasting error

The price parameter $b^*_t$ in (14) and the conditional variance terms in (16) remain interdependent. The rational expectations equilibrium, based on the values of $b^*_t$, $\sigma^2_M$ and $\sigma^2_F$ that solve (14)-(16), though complex, does exist. Without knowledge of $n$, the traders, though, are unable to deduce the $n$ dependent endogenous values such as $\sigma^2_M$ and $\sigma^2_F$. Further, away from the fixed point equilibrium, as is the case during the learning process, knowledge of $c_{it}$ is insufficient to determine $\sigma^2_M$ and $\sigma^2_F$ without knowledge of $n$. These parameters, too, must be either estimated or, employing bounded rationality, presumed by the traders to take certain values.

With $\sigma^2_M$ and $\sigma^2_F$ in the denominator of the demand function, how the traders choose to estimate the values of these parameters affects the market. Four reasonable strategies by which traders might estimate $\sigma^2_M$ and $\sigma^2_F$ are considered. All four generate a fixed point to the learning process. Differences in the fixed point exist, but are largely inconsequential. More important is the impact on convergence.

**Case I:** Impose rational expectations conditional on $c_{it}$, implying that the traders know, or in aggregate behave as if they know, the conditional variance terms as they depend on $c_{it}$ and $n$ despite the lack of access to $n$. The importance of the rational expectations approach is that the current market environment is incorporated into the estimates according to (13). In particular,

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3 Solving for $b^*_t(n)$ involves solving a cubic polynomial. The unique real solution is available, but pages long.
\( \sigma^* \) increases as the error in \( c_t \) increases. This has a stabilizing effect on the market, ensuring a well behaved price. Numerical examination reveals that \( \Psi(n, c_t) > 0 \) holds for all parameter combinations. The price remains finite for all values of \( c_t \) without bound.

Case II: Following Brock and Hommes (1998) and much of the related literature, the traders presume that \( \sigma^2_M = \sigma^2_P \). Conceptually, this approach exploits the fact that the presence of \( \sigma^2_P \) and \( \sigma^2_M \) in the price solution is the incorporation of \( \sigma^2_P \) and \( \sigma^2_M \) by the investors in their demand function. The traders’ presumption of \( \sigma^2_M = \sigma^2_P \) eliminates the variance terms from the price solution in (14). This presumption result is a substantial simplification of the solution, making the calculation reasonably accessible to the trading population. To the detriment of market functionality, \( \Psi(n, c_t) \leq 0 \) becomes possible. The upper bound \( \overline{\tau}(n) \) plotted in Figure 1 and included in each of the simulation outputs is obtained by solving for \( c_t \) as a function of \( n \) (or \( n_t \)) using \( \Psi(n, c_t) = 0 \) based on \( \theta^I = \theta^F \).

Case III: Following Goldbaum (2003), traders use data generated by the market to empirically estimate \( \hat{\sigma}^2_P \) and \( \hat{\sigma}^2_M \). The result is that the period \( t \) estimates, \( \hat{\sigma}^2_P \) and \( \hat{\sigma}^2_M \), are fixed at time \( t \) as determined by the accumulated data gathered through time \( t - 1 \). Combination of \( c_t \) and \( n \) that produce \( \Psi(n, c_t) \leq 0 \) remain, ensuring the existence of a some upper-bound on \( c_t \) to keep the price finite. The value of the upper bound depends on \( \hat{\sigma}^2_P \) and \( \hat{\sigma}^2_M \) so that the upper bond generally will not match \( \overline{\tau}(n) \) of Case II.

Case IV: The market-based traders include the current price innovation in their estimate of \( \hat{\sigma}^2_M \), making demand no longer linear in price. The substantial demand induced among the market-based traders by a large price increase is tempered by the realization that the estimate implies
considerable uncertainty. Having $\hat{\sigma}_{M}^2$ determined in conjunction with the concurrent price ensures $\Psi(n,c_t) > 0$ so that the price remains bounded regardless of the error in $c_t$. Like Case I, there is no upper bound on the value of $c_t$ required to keep the price finite. The benefits of the rational expectations price solution is approximated without relying on high level sophistication by the traders nor unreasonable access to the knowledge of the strategies employed by the other traders.

By endogenizing the estimates of $\sigma_{M}^2$ to the current price, Case I and Case IV both ensure a finite market clearing equilibrium price always exists, though the price may still exist outside of the $[p_t^F, p_t^{EM}]$ range. Due to the impact of the concurrent price on the trader’s estimate of $\sigma_{M}^2$, the price deviation produced in Case IV will always be less than that produced in Case III, all else equal.

The time $t$ fixed estimates of $\sigma_{F}^2$ and $\sigma_{M}^2$ in both Case II and Case III make possible a realization of $c_t$ that is incompatible with $n$ in such a way as to produce either an infinite market clearing price as $c_t \rightarrow \overline{c}_1^*$ or a non-existent price if $c_t > \overline{c}_1^*$.

3.3 Market Efficiency

Let $p_t^{EM}$ be the standard against which the market price is evaluated. Let $|p_t - p_t^{EM}|$ be the measure of market efficiency. In general

$$p_t - p_t^{EM} = (b_1 + \phi b_{2t} - \phi/(R-\phi))\eta_t + (\phi b_{2t} - \phi/(R-\phi))\epsilon_{t+1}.$$  \hspace{1cm} (18)

From (18), there are two sources of deviation from $p_t^{EM}$. At the learning fixed point, $b_1'(n) + \phi b_2'(n) = \phi/(R-\phi) \forall n$ and $b_2'(n) \rightarrow 1/(R-\phi)$ as $n \rightarrow 0$. Thus, the first term of (18) is
zero when the market-based model is correct, \( c_i = c_i^* \), so that \( p_i = p_i^* \). When the market traders have correct beliefs, the second term approaches zero as the bias caused by the fundamental traders becomes negligible so that \( p_i^* \to p_i^{EM} \) as \( n \to 0 \).

3.4 Performance

The two sources of error in (18) determine relative performance between the different strategies. When \( c_{it} \approx c_i^*(n) \) the market traders’ model is close to correct and \( p_i \approx p_i^* \). In this case, the deviation from \( p_i^{EM} \) benefits the market traders as the deviation is the result of the bias in the demand of the fundamental traders. When the model extraction error is large, this benefits the fundamental traders as the price error is dominated by the impact of the market traders’ error.

Formally, let the measure of profits earned by each information source be the excess return realized for the risky asset multiplied by the group average demand:

\[
\pi_i^k = q_i^k \left( p_{i+1} + d_{i+1} - R_{p_i} \right), k = F, M. \tag{19}
\]

Based on \( c_{it} = c_i^*(n) \), expected profits reduce to,

\[
\begin{align*}
E(\pi^{*F}) &= -n(1-n)\Delta^* \\
E(\pi^{*M}) &= n^2\Delta^*
\end{align*}
\]

for \( 0 < n \leq 1 \) \tag{20}

where

\[
\Delta = \theta^{\epsilon_2} \theta^M \left( \frac{R}{R-\phi} \right)^2 \left( \frac{(1-\beta)^2}{(n\theta^F + (1-n)\theta^M)^2} \right) \sigma^2_e
\]

and \( \Delta^* \) is \( \Delta \) evaluated with \( \sigma_F^2 = \sigma_F^{*2} \) and \( \sigma_M^2 = \sigma_M^{*2} \).

With a correct model of the relationship between price and payoff, the fully revealing price is a better source of information than the noisy signal for \( n < 0 \). Appropriately, the market-based traders outperform the fundamental traders, as reflected in (20).
For $c_i \neq c_i^*(n)$, the market-based strategy can lose its information advantage depending on the size of the parameter error and on $n$. The market-based model will generate profits for $c_i^- < c_u < c_i^+$. Within this range, the model error is sufficiently small so that the information extracted from the price, even with error, still dominates the information derived with error from individual fundamental research. As captured in Figure 1, the ability of the market to tolerate error diminishes as $n$ declines. A given error in the model will have a greater market impact the fewer fundamental traders there are to stabilize the price. The result is that the range in $c_u$ for which the regression model earns profit shrinks with $n$.

3. A population process

As GS point out, it is reasonable to think that traders would respond to differences in performance. It is reasonable to consider modeling evolution in $n_t$ over time based on performance difference in the two strategies. As will be demonstrated, the leaning process that traders might reasonably choose to employ in the absence of knowledge of $n_t$ becomes a source of instability in the presence of an evolving population.

3.5 A dynamic process for the population

Innovation to the population’s level of reliance on the fundamental and market-based information reflects the traders’ perception of which of the two approaches offers a trading advantage. As with other endogenously determined parameters, since $E(\pi^*_E)$ and $E(\pi^*_M)$ are dependent on $n_t$. Without knowledge of $n_t$, the traders will not know their values. Let $\hat{\pi}_t^E$ and $\hat{\pi}_t^M$ indicate the traders’ performance measures of fundamental and market-based approaches, respectively. These are updated according to the process
\[
\hat{\pi}_t^k = \hat{\pi}_{t-1}^k + \mu_t (\hat{\pi}_{t-1}^k - \hat{\pi}_{t-1}^\ell),
\]

(21)

\[
\hat{\pi}_0^k = 0, k = F, M. \text{ The measures are thus based on past performance.}
\]

The Replicator Dynamic produces an evolutionary dynamic population in which the dominant strategy attracts converts from the inferior strategy. The two choice version of the more general \(K\) choice replicator dynamic model found in Branch and McGough (2005) results in the transition equation

\[
n_{t+1} = \begin{cases} 
n_t + r(\hat{\pi}_t^F - \hat{\pi}_t^M)(1 - n_t) & \text{for } \hat{\pi}_t^F \geq \hat{\pi}_t^M \\
n_t + r(\hat{\pi}_t^F - \hat{\pi}_t^M)n_t & \text{for } \hat{\pi}_t^F < \hat{\pi}_t^M 
\end{cases}
\]

(22)

A number of different functional forms for \(r\) exist in the literature. The simulations that follow are based upon

\[
r(x) = \tanh(\delta x / 2)),
\]

(23)
a choice that ensures \(0 < m < 1\) for bounded \(\hat{\pi}_t^F - \hat{\pi}_t^M\). By construction, the discontinuity at \(n = 0\) will never be realized in the simulation. The parameter \(\delta\) determines how responsive the population is to differences in expected profits.

**Proposition 3**: Given a fixed \(c_1\), there is a fixed point to the population process, \(n^*(c_1)\) that generates \(E(\pi_t^F) = E(\pi_t^M)\) and at which \(c_1 > c_1^*(n^*)\).

**Proof**: For \(c_1 \neq c_1^*\), both groups of traders estimate the \(d_{t+1}\) with error. A \(n^*\) exists, \(n^*(c_1) \in (0,1]\). Both \(E(\pi_t^F)\) and \(E(\pi_t^M)\) are smooth monotonic function in \(n\) with \(E(\pi_t^F)\) increasing and \(E(\pi_t^M)\) decreasing as \(n\) decreases. \(n = 1\) favors the market-based investors regardless of the value of \(c_1\) while \(n \to 0\) generates \(E(\pi_t^F) \to \infty\) for \(c_1 \neq c_1^*\), producing a single crossing.

For \(n > 0\), there is no combination of \(c_1, n\) at which both the learning process and the population process are at equilibrium. For \(c_1 = c_1^*\), \(E(\pi^M) > E(\pi^F)\) so that \(n_t\) declines. As in GS, \(n_t = 0\) cannot be a fixed point and thus for \(c_1 = c_1^*\), there is no fixed point to the \(n_t\) process.
Likewise, if \( n = n^*(c_i) \), then \( c_i \neq c_i^* \).

The absence of a system wide fixed point, \((c_i^*, n^*)\), undermines the notion of a rational expectations equilibrium. Should such a fixed point exist, it could be deduced by the rational traders allowing the formation of a REE. In the absence of such a fixed point to the population process, the trader cannot presume to know \( n_t \) which, in this setting means they cannot know the values of \( c_i^*(n^*) \) or \( b^*(n^*) \). A REE is not feasible and thus traders must resort empirical analysis and learning to gain an understanding of the market they populate.

### 3.6 Simulations

Simulations help to characterize the dynamic processes in the absence of an equilibrium. When the market-based strategy performs poorly, the result of self-induced mispricing, \( n_t \) increases towards 1. The increased proportion of fundamental traders adds to the stability of the price, a self-corrective market mechanism. An accurate market-based model allows the market-based traders to exploit the information content of the price, earning profits so that \( n_t \) declines. The narrowing of the profitable range of regression parameters that follows from a decrease in \( n_t \) means that a decline in \( n_t \) is sustainable only if the market-based model is sufficiently accurate to support the decline in \( n_t \). A slow and smooth decline in \( n_t \) ensures that the rate of learning is “faster” than the rate of transition in the population. “Faster” learning can be defined as the situation in which the distribution of \( c_{it} \) narrows faster than does the distance between \( c_i^- \) and \( c_i^+ \) so the probability that \( c_{it} \notin [c_i^-, c_i^+] \) decreases with time and with the evolution of the market. With this comes the decrease probability that a wildly inaccurate price will be realized.

Two parameters influence the rate and smoothness of the evolution in \( n_t \). A large \( \delta \) in (22) means that traders are more responsive to differences in performance, producing large shifts in \( n_t \).
in response to small performance differences. The parameter \( \mu_i \) in (21) determines the emphasis placed on the current realizations of profits relative to the full simulation history when determining the performance measure.

The rate at which beliefs evolve is determined by \( \lambda_i \). If the least-squares model is employed, with \( \lambda_i = 1/t \), then learning slows as \( t \) increases. From Proposition 1, there is a lower bound in the accuracy of the market-based model when constant gain is employed.

Figures 2 through 7 display the typical evolution of endogenous parameters produced by the simulations. Each figure is based on the same underlying \( d_i \) series. Each frame plots the time progression of one of the endogenous parameters of the model. Across the top row are plotted the price parameters \( b_{1i} + \phi b_{2i} \) and \( \phi b_{2i} \), each determining the coefficients of the two terms in (18). The time series are plotted in green. Both frames include a solid black line at \( \phi/(R-\phi) \).

The second row of frames contains the regression coefficients from the market traders’ model, \( c_{1i} \) and \( c_{2i} \). In simulations, \( c_{1i} \) and \( c_{2i} \) are decoupled, not constrained to satisfy (15). The value of \( c_0 \) is presumed to be known. The red time-series plots \( c_i^*(n_i) \) for the current \( n_i \), while the black line plots the respective element of \( c^* \) \( |_{n \to n_0} = c^- \). Also included with \( c_{1u} \) is \( \overline{c}_{1u} \) based on Case II plotted in blue.

The bottom row presents the population parameter \( n_i \) and the difference \( p_i - p_i^{CM} \).

All simulations employ \( R = 1.02 \) and \( \sigma_c^2 = \sigma^2 = 1 \), the latter producing \( \beta = 0.5 \). All of the simulations were examined based on traders forming estimated conditional variances as in Case III and Case IV. Case III, with the upper bound on \( c_{1u} \) of \( \overline{c}_{1u} \), is more prone to realizations of extreme price deviations. The discussions of Simulations 1 and 2 are based on the Case III
environment. The discussions of the remaining simulations are based on Case IV.

**Simulation 1:** Slow steady convergence

Simulation 1 highlights the behavior of the market under conditions conducive to a well behaved price. Learning is via the least-squared process of (17) with $\lambda_t = 1/t$. In estimating the model, the trader uses all the available information, giving equal weight to each observation. The same is true for computing performance, with $\mu_t = 1/t$. The population is set to respond slowly to the performance differential with $\delta = 0.01$. The result of the simulation under Case III is captured in Figure 2, which plots the last 50,000 observations of the $T=500,000$ simulation.

The simulation maintains a slow and steady decline in $n_t$ that results in a slow and steady decline in $c_t^\ast$. The estimated $c_t$ fluctuates in a narrow range around $c_t^\ast$. The generous distance between $c_t^\ast$ and $c_t^-$ relative to the distribution of $c_t$ suggests the market could support an even larger proportion of market based traders, but this need not be the case. The rate of decline in $n_t$ is tempered by the accuracy of the market-based model as Simulation 2 will make apparent.

Since $c_t$ never approaches $c_t^\ast$, the simulation is unaffected by whether estimation is based on Case III or Case IV.

Relative to the simulations to follow, the price deviations from efficiency are small. The magnitude of the error does display clustering, reflecting the persistence in the regression parameter deviations from their correct value. The average pricing error remains constant as the result of the balanced co-evolution in learning and the population.

**Simulation 2:** High sensitivity to performance

In Simulation 2, $\delta = 10$, creating a substantially more responsive population generates cycles in $n_t$ on top of its underlying process of decline. When $c_t$ is accurate, the market-based trades
perform well and \( n_t \) quickly declines. With the decline in \( n_t \), \( c_{it} \) is no longer accurate for the current state suffering from an upward bias. The error benefits the fundamental traders, increasing \( n_t \). This, in turn, increases price accuracy and the ability of the market-based strategy to profit. The cycles can be seen in Figure 3. Though \( \bar{c}_{it} \) fluctuates with \( n_t \), it never drops below the relatively stable \( c_{it}^{+} \). This is because even though the swings in \( n_t \) are large, the process is smooth. Since \( c_{it}^{+} < \bar{c}_{it} \) also fluctuates with \( n_t \), once \( c_{it}^{+} \) (not shown) crosses below \( c_{it}^{+} \), the \( n_t \) process reverse its decline. The distribution in the pricing error cycles with the other parameters of the model. As in Simulation I, even though conditional variance are determined as in Case III, price spikes do not arise during the cycles in \( n_t \) and \( c_{it}^{+} \).

**Simulation 3:** Adaptive expectations in performance

In simulation 3 traders employ adaptive expectations when determining performance, with \( \mu_{i} = 1 \). In addition, \( \delta = 0.1 \) to produces a medium rate of adjustment.

The market characterized in this simulations is one in which market-trader beliefs are generally consistent with the prevailing market. The short memory in performance produces a noisy \( n_t \) process. On rare occasion, the market shifts so that these same beliefs lead to market failures. With traders responding to the most recent realization of performance, relatively large discrete jumps in \( n_t \) do occur, which can be seen in Figure 4.

The most striking feature of the simulation is that adaptive expectations in the population halts the convergence in learning and thereby halts the decline in \( n_t \).

There is a feedback so that the volatility in the \( n_t \) process produces and is sustained by the volatility in \( \bar{c}_{it} \). Figure 4 reveals that the \( \bar{c}_{it} \) series experiences a number of realization that are close to or below the more stable \( c_{it}^{+} \). The price remains finite in the latter case because the
simulation is examined under Case IV. Still, the spikes in the price are extreme so that the market is characterized by reasonably stable price fluctuations punctuated with short-lived spikes of substantial error. These events are sufficient to prevent convergence in learning.

Figure 5 plots a short 50 periods, starting at \( t = 492300 \), during which one of these spikes occurs. Just before the spike, there is a period of decline in \( n_t \) that bring \( \pi_t \) close to the prevailing \( c_t \). Using just the last two digits to refer to the period number, a series of unusually large innovations in dividends in periods \( t=35, 36 \) and particularly large innovations in \( t=38, 39 \) generate high returns for the market based traders, producing the observed decline in \( n_t \). From \( t=30 \) to \( t=39 \), \( n_t \) drops from 0.366 to 0.308. In \( t=40 \), \( n_t \) is at 0.221, a value too low given the accuracy of the market model. A negative dividend innovation in this period causes the asset to be substantially under priced by about 10 price units relative to the efficient market price. The price recovers the following period and profits from this mispricing are realized. The realized profit enters \( \hat{\pi}_t \) in period 42 so that in period 43 \( n_t \) jumps to 0.942. It takes about 150 periods before \( n_t \) is below 0.5 again.

If \( \delta \) is decreased, for example to \( \delta = 0.01 \), then the jumps in \( n_t \) are too small to bring \( \pi_t \) close to the prevailing \( c_t \), so the price spikes are never realized. Still, the convergence in learning is halted by the short memory in the population process with \( n_t \) settling into a narrow distribution at about 0.4.

**Simulation 4**: Simulation 4 switches the model updating from least-squares learning to a constant gain process with \( \lambda \in (0,1] \) and returns to \( \mu_t = 1/t \). At \( \lambda = 0.01 \), memory is not short, but is still finite and weighted towards more resent observations. Figure 6 displays the final 200,000 periods of the simulation.

For the majority of the time, the market is well behaved, generating small but steady profits for
the market-based strategy. Dissruptions to the market are rare, but inevitable. The profits to the
market-based strategy leads to a steady decline in $n_t$. With the declined in $n_t$ comes a narrowing
of the market’s tolerance for error in the regression model coefficients. This is captured by the
decline in $\bar{\tau}_{\mu}$ in Figure 6. The need for accuracy in the regression model cannot be delivered.
Consistent with Proposition 2.2, the distribution of $c_{\mu}$ around $c^{*}_{\mu}$ appears to stabilize into a fixed
distribution. The encroachment of $\bar{\tau}_{\mu}$ into the distribution space for which $c_{\mu}$ enjoys high
probability of realization leads to sudden and dramatic increase in both the pricing error and the
noise in the regression coefficients.

The large profits earned by the fundamental traders during the brief period of mispricing
and the long memory in performance ensure $\hat{\pi}_{\mu}^{F} > \hat{\pi}_{\mu}^{M}$ long after the normalcy favoring the
market traders has returned to the market. This, in turn, sustains the long rise in $n_t$ that
progresses slowly due to the low $\delta$ setting.

For the majority of the time, the market-based strategy generates small but regular profits.
On rare occasions, the market requires a great deal of accuracy in the regression model employed
by the market-based traders, but because of the short memory employed in estimating the model,
this accuracy cannot be consistently produced. The result is substantial mispricing that offers
short lived but substantial profits to the fundamental strategy.

**Simulation 5**: Simulation 5 incorporates finite memory in both processes with $\mu_\tau = 1$ and $\lambda = 0.01$.

As seen in Figure 7, The noise in both the population process and the regression
coefficients again combine to produce sustained periods of a reasonable well behaved market
with spikes of varying magnitude generated by realizations of $c_{\mu}$ near to $\bar{\tau}_{\mu}$.
4. Conclusion

As simple and reasonable as is the underlying market employed in this analysis, a rational expectations equilibrium cannot be presumed. Instead, the environment requires that traders form beliefs about price formation and the profitability of different strategies based on market observation and experience. Extreme market prices and even the absence of a market clearing price are the product of trader efforts to learn and adapt to the changing conditions that arise in the market they populate. That the changing market condition is the product of market evolution perpetuates the problem.

A slow and measured response by market participants to new information creates the stability necessary for the trades to develop an accurate understanding of the price formation process. It is common practice, though, for traders to place greater weight on more recent observations in order to optimize their strategy to current market conditions. This short memory is found to contribute to the instability to which the traders are trying to adapt. Short memory makes inevitable excessive reliance on an imperfect model that results in substantial pricing error.

References


Figure 1: Phase Space in $n_t$ and $c_{1t}$. 
Figure 2: Slow and smooth evolution in population and learning; \( \delta = 0.01, \mu = 1/t, \lambda = 1/t, t = 450,000 \text{ to } 500,000 \)
Figure 3: Enhanced sensitivity to performance difference; $\delta = 10$, $\mu_r = 1/t$, $\lambda_r = 1/t$, $t = 450,000-500,000$. 
Figure 4: Adaptive expectations applied to performance, $\delta = 0.1$, $\mu_t = 1$, $\lambda_t = 1/t$, $t = \text{450,000-500,000}$
Figure 5: Adaptive expectations applied to performance (sub-sample), $\delta = 0.1$, $\mu_i = 1$, $\lambda_i = 1/t$, $t = 492,300-492,350$
Figure 6: Constant gain in learning; $\delta = 0.01$, $\mu = 1/t$, $\lambda = 0.01$, $t = 300,000-500,000$. 
Figure 7: Adaptive expectations in performance and constant gain in learning; $\delta = 0.01$, $\mu_i = 1$, $\lambda = 0.01$, $t = 450,000-500,000$. 