Learning and Adaptation as a Source of Market Failure*

David Goldbaum†

April 3, 2013

Abstract

Financial market participants learn from and adapt to an evolving market environment. In the developed model, intuitively appealing and extensively employed mechanisms of adaptation can produce episodes of extreme misplacing in the absence of a rational expectation equilibrium. The destabilization of the market originates from empirically supported over-reliance on market-based information. Increasing the facilities of the fundamental traders to increase the rationality of their strategy can exacerbate the mispricing.

Keywords: Heterogeneous Agents, Efficient Markets, Learning, Dynamics, Computational Economics, Market Failure
(JEL Codes: G14, C62, D82)

1 Introduction

An extraordinary number of traders employing a wide variety of strategies populate financial markets. Many attempt to extract rent through trading. Vigorous trading and extensive market commentary suggests a lack of uniformity among market participants and possible disagreement as to the true price determination process. The disagreement extends to issues of market efficiency and how possible deviations from efficiency can best be exploited. The diversity in trading strategies spans value seeking based on fundamental analysis to highly complex efforts to extract tradable information from the markets to efforts to exploit anomalies.1 Within these categories, individual decisions regarding the processing of information contribute to the large diversity of beliefs and strategies.

---

*I am indebted to J. Barkley Rosser for providing inspiration for the project and the Paul Woolley Centre for Capital Market Dysfunctionality for financial support.
†Economics Discipline Group, University of Technology Sydney, PO Box 123 Broadway, NSW 2007 Australia, david.goldbaum@uts.edu.au
1Labels applied to trading strategies such as fundamental trading, speculation, chartism, and technical analysis capture this diversity.
The model employed in the current investigation is derived from Goldbaum (2006), a financial market that produces a market clearing price, but not a rational expectations equilibrium (REE). The lack of a REE is a result of the market information, consisting of individual idiosyncratically noisy private signals on the future dividend, and trader profit seeking behavior. In the absence of a REE fixed point, data-driven adaptation and learning are reasonable trader accommodations to uncertainty regarding the price determination process, leading to ever-changing market conditions. This analysis focuses on the potential that trader accommodations to their environment have the potential to be disruptive to the market.

This investigation explores the consequences bounded rationality on market behavior. The case will be made that the model offers attractive features absent in other adaptive agent-based financial market models, providing insight into markets not otherwise available. Goldbaum (2006) provides the platform for examining adaptation and learning by financial market participants. The distinguishing feature of this model is that the extreme market events arise from individual trader learning and adaptation to their environment rather than the product of an inherently destabilizing market-based strategy, thereby offering insights into the nature of market disfunctionality absent from other adaptive agent-based models.

Financial markets have long been recognized as potential feedback systems between market behavior and trader beliefs. Beja and Goldman (1980) formally incorporated such feedbacks. The complexities associated with devising an optimal trading strategy in the presence of market feedback can lead to considerable uncertainty for the trader. This uncertainty can be incorporated into a model as heterogeneity and bounded rationality. Frankel and Froot (1990) and De Long, Shleifer, Summers, and Waldmann (1990a) and De Long, Shleifer, Summers, and Waldmann (1990b) define different groups of traders in order to formally explore the impact of trader heterogeneity on the markets.

Statistical learning is one of a number of mechanisms developed to allow boundedly rational agents to adapt to their environment within the limitations of their information or ability. A learning process can generate emergence of rational behavior despite the limits on the agents’ information or sophistication, thereby achieving the REE without necessarily relying on fully rational agents.
Marcet and Sargent (1989a), Marcet and Sargent (1989b) and Evans and Honkapohja (2001) include demonstrations of such convergence as examples in the formal development of least-squares learning in. Examples include learning in the models of Bray (1982) and Townsend (1983). In contrast, Bullard (1994), Bullard and Duffy (2001), and Chiarella and He (2003) highlight the possibility of non-convergence, as determined by the conditions under which the learning takes place. The learning model of Timmermann (1996) also has the REE as the asymptotic limit but the author observes that the path dependent effects of learning can be quite long lived, thus generating persistent non-equilibrium behavior.

Forms of adaptation distinct from learning have also been explored. In LeBaron, Arthur, and Palmer (1999) initiate each trader with a unique portfolio of trading strategies that are updated according to a genetic algorithm allowing the trader to evaluate, switch, disassemble and reassemble the trading rules employed. Also prevalent in the literature are more structured models of switching over a fixed set of universal strategic alternatives. Of interest in switching models is how a strategy based on market fundamentals or rational expectations performs in comparison to, and in the presence of, alternative trading strategies. Non-rational strategies are demonstrated to survive and even prosper in such competitive settings. The feedback between the market and beliefs can be particularly important in the earned profits and population adoption of alternatives to fundamental strategies.

The two-choice model often consists of a fundamental strategy and a market-based strategy, such as this example from Gaunersdorfer and Hommes (2007),

\[
E_{1t}(p_{t+1}) = p_{1,t+1} = p^* + v(p_{t-1} - p^*), \quad 0 \leq v < 1, \\
E_{2t}(p_{t+1}) = p_{2,t+1} = p_{t-1} + g(p_{t-1} - p_{t-2}), \quad g \geq 0,
\]

where \(p^*\) represents the fundamental value.\(^3\) The fundamental value drives expectations in the first equation while the alternate extrapolates the future value from current price innovation. The

\(^2\)Also see Sargent (1993).

\(^3\)This formula is also employed by Frankel and Froot (1987). Variants of this trend following and fundamental reverting system are common in two choice models.
alternate is thus trend-following in nature.\(^4\)

In a dynamic setting the fundamental model is inherently stabilizing to the system so that, when dominate, the market tends towards its correct well-behaved fundamental value. The trend-following strategy is inherently destabilizing in that it tends to cause the market to drift away from its fundamental value.

Cycles or complex dynamics of convergence and divergence can emerge as a result of the switching mechanism employed in the model. This is the particularly true for models in which the population shifts towards adoption of trend-following expectations when the market is near the REE and towards fundamental expectations when the market is far from equilibrium. Such population dynamics are explicit in De Grauwe, Dewachter, and Embrechts (1993), Giardina and Bouchaud (2003) and Lux (1998) while achieved indirectly as the product of performance chasing in Brock and Hommes (1998) and Brock and LeBaron (1996), among others.\(^5\)

The dynamics of performance driven models are fueled by the contradiction inherent in the environment. Near the market fundamental all trading generates zero profit so that the fundamental investor is unable to recover the cost of the fundamental strategy through trade but the absence of profits at the REE alone is insufficient to induce abandonment. Typical is the imposition of a cost on the fundamental information, a feature motivated by the greater level of sophistication necessary to implement a fundamental strategy. The trend following rule, the only cost-saving alternative available, conveniently has the potential to generate the very market trend that fuels its own profits with increasing adoption by the trading population. In this case, the success of the rule is self-propagating and divorced from the market fundamentals.

Brock and LeBaron (1996) provides the fundamental traders with useful but idiosyncratically noisy private information. In contrast, Grossman and Stiglitz (1980) provides uniform error-free information and allows the market-based traders access to the current price, providing a mecha-

\(^4\)Sometimes a contrarian strategy is incorporated in place of or in addition to the trend-following strategy, for example in Brock and Hommes (1998) and Chiarella and He (2001). Bias in the form of optimism and pessimism is another form of heterogeneity as employed by Lux (1995)Lux (1998) and Kirman (1993). Brock and Hommes (1997) allows for bias and extrapolation.

\(^5\)Hommes (2006)provides a summary of related dynamic heterogeneous agent models. Past performance can also have an impact on the influence of a strategy on the market through wealth effects, as in Chiarella and He (2001), Farmer and Joshi (2002), Chiarella, Dieci, and Gardini (2006), and Sciubba (2005).
nism for the trend-followers to extract the private information. The combination of the Brock and LeBaron noisy private signal and the Grossman and Stigliz contemporaneous market information results in a market in which the price can be a superior source of information to the private information which allows Goldbaum (2003) to eliminate the arbitrary information cost as a necessary mechanism to motivate trader abandonment of the fundamental-based strategy.

Goldbaum (2005) and Goldbaum (2006) substitutes a least-squares learning model for the fixed trend-following rule. The result is a market-based trading strategy that is not inherently destabilizing. The combination of the dynamics of learning and switching is the alternative source for financial market features such as clustered volatility and fat-tailed returns. Together, the learning and the use of the replicator dynamic (RD)\textsuperscript{6} to capture trader choice create a Grossman and Stiglitz paradox with the accompanying absence of an equilibrium fixed point.

From both Brock and LeBaron (1996) and LeBaron, Arthur, and Palmer (1999) long memory stabilizes the dynamic system by encouraging a focus on fundamentals while short memory encourages short-run profitable behavior that is destabilizing to the system. Goldbaum (2006) considers only long-run properties of the market employing convergence friendly parameters to produce a non-equilibrium environment that is asymptotically stable.

An exploration of learning and adaptation can also be found in Branch and Evans (2006), but their use of the discrete choice dynamics (DCD)\textsuperscript{7} with long memory ensures the existence of a fixed point. At the Branch and Evans fixed point, there is no longer interaction between the learning and population processes.\textsuperscript{8} This is in contrast to Goldbaum (2006) where the nature of the interaction is the source of the asymptotic stability.

The absence of a rational expectations equilibrium makes the Goldbaum (2006) model a useful environment in which to explore the impact of adaptation and learning on market stability. That is, the model contains a tension between those relying on imperfect fundamental information and those seeking to optimally exploit the information content of market phenomenon. Neither strategy

\textsuperscript{6}Employed by Sethi and Franke (1995), difference in performance leads to population shift towards the more successful strategy.

\textsuperscript{7}Employed by Brock and Hommes (1997). the population is divided between strategies so that the more successful strategy is employed by a greater proportion of the population.

\textsuperscript{8}Learning and adaptation can be considered present in the process of individual trading rule adjustment and discovery produced by the genetic algorithm employed by LeBaron, Arthur, and Palmer (1999).
is inherently superior so that there is no need to introduce an arbitrary cost. Further, the market based alternative to the fundamental information is not inherently destabilizing but contributes towards market efficiency when used appropriately.

The current project explores how learning and strategy switching as a form of adaptation, in the presence of commonly employed constrains on trader rationality, can combine to disrupt the market to produce extreme price events.

2 Model

The dynamic model, developed from the market environment originally proposed in Goldbaum (2006), is defined by the following set of linear and non-linear transition equations:

\[
\begin{align*}
    d_{t+1} &= \phi d_t + \epsilon_{t+1} \\
    p_t &= p_t(n_t, c_t) = b_0 + b_1(n_t, c_t)d_t + b_2(n_t, c_t)d_{t+1} \\
    c_t &= c_{t-1} + \lambda_t(Q_{t-1}x_{t-2}(p_{t-1} + d_{t-1} - c_{t-1}x_{t-2}))' \\
    Q_t &= Q_{t-1} + \lambda_t(x_{t-1}x_{t-1} - Q_{t-1}) \\
    \hat{\sigma}_{kt}^2 &= \hat{\sigma}_{kt-1}^2 + ((z_t - E(z_t|Z_{t-1}))^2 - \hat{\sigma}_{kt-1}^2)/t \\
    \hat{\pi}_t^k &= \hat{\pi}_{t-1}^k + \mu_t(\pi_{t-1}^k - \hat{\pi}_{t-1}^k), k = F, M \\
    n_{t+1} &= f(\hat{\pi}_t^F - \hat{\pi}_t^M, n_t)
\end{align*}
\]

where \( f(x, n) \) is a monotonically increasing function in \( x \) with \( 0 \leq f(x, n) \leq 1 \).

This project makes a case for the use of model-consistent limits on trader rationality and investigates the impact of these limitations on market behavior.

2.1 Model

The market environment consists of a risky dividend-paying asset and a risk-free bond paying \( R \). The risky asset can be purchased at price \( p_t \) and is subsequently sold at price \( p_{t+1} \) after paying the holder dividend \( d_{t+1} \). Equation (1) defines the exogenous dividend as AR(1) with
innovations distributed $\epsilon_t \sim \text{IIDN}(0, \sigma^2_\epsilon)$. The traders select one of two forecast methods based on the information sets

$$Z_{it} \in \{Z^F_{it}, Z^M_{it}\}$$

$$Z^F_{it} = \{s_{it}, d_t, d_{t-1}, \ldots\}$$

$$Z^M_{it} = Z^M_t = \{p_t, d_t, d_{t-1}, \ldots\}.$$  

All traders have access to the dividend history up to time $t$. The fundamental method (F) consists of a noisy signal on next period’s dividend,

$$s_{it} = d_{t+1} + \epsilon_{it} \quad (8)$$

$$\epsilon_{it} \sim \text{IIDN}(0, \sigma^2_\epsilon).$$

The market-based approach (M) makes use of endogenous market-generated information. Consistent with the model, the market-based information need consist of $p_t$ as the only endogenous information source.

The presence of $d_{t+1}$ in the price equation reflects the demand and knowledge of the fundamental traders through their private signal. Fundamental trader uncertainty, the consequence of knowing the idiosyncratic component of their signal, explains the presence of $d_t$. Fundamental traders minimize the prediction error when projecting, $E(d_{t+1}|Z^F_{it}) = (1-\beta)\phi d_t + \beta s_{it}$ with $\beta = \sigma^2_\epsilon/((\sigma^2_\epsilon + \sigma^2_\epsilon)$ as the signal to noise ratio. So long as the fundamental traders forecast of $p_{t+1}$ is consistent with the structure

$$E(p_{t+1}|Z^F_{it}) = E\left( \frac{1}{R - \phi} \left((1-x)\phi d_{t+1} + xd_{t+2}\right) \right)$$

for any $x \in [0, 1]$, then

$$E(p_{t+1} + d_{t+1}|Z^F_{it}) = \left( \frac{R}{R - \phi} \right) E(d_{t+1}|Z^F_{it}) .$$

The particular value of $x$ drops out of the fundamental model’s forecast of the uncertain payoff. This is convenient from a modeling perspective as the market clearing price will not require specifying

\[A value x = \beta is consistent with a F-trader only market clearing price.\]
a belief about $x$.

The market-based traders employ a forecasting model that includes all relevant variable consistent for forecasting the following period’s payoff,

$$E(p_{t+1} + d_{t+1}|Z_t^M) = c_0 + c_1 p_t + c_2 d_t$$ (11)

Traders submit a demand function to maximize a CARA utility function with risk aversion coefficient $\gamma$, and $\sigma^2_{kt} = \text{Var}_it(p_{t+1} + d_{t+1}|Z_{it}^k)$,

$$q_{it}(p_t) = \frac{(E(p_{t+1} + d_{t+1}|Z_{it}) - Rp_t)/\gamma \sigma^2_{kt}}{\gamma \sigma^2_{kt}}.$$ (12)

The forecast errors for the two models are

$$p_{t+1} + d_{t+1} - E(p_{t+1} + d_{t+1}|Z_{it}^F) = \phi \left( 1 + b_{1t} + \phi b_{2t} - \frac{R}{R-\phi} \right) d_t + \left( 1 + b_{1t} + \phi b_{2t} - \frac{R}{R-\phi} \right) \epsilon_{t+1} + b_{2t} \epsilon_{t+2} + \beta \frac{R}{R-\phi} \sigma^2_{it}$$ (13)

$$p_{t+1} + d_{t+1} - E(p_{t+1} + d_{t+1}|Z_{it}^M) = (\phi(1 + b_{1t} + \phi b_{2t}) - c_{1t}(b_{1t} + \phi b_{2t}) - c_{2t})d_t + (1 + b_{1t} + \phi b_{2t} - c_{1t}b_{2t})\epsilon_{t+1} + b_{2t} \epsilon_{t+2}.$$ (14)

With $N_{nt}$ traders using the fundamental approach and $N(1 - n_t)$ employing the market-based approach, the Walrasian market clearing coefficients of (2) are

$$b_0(n_t, c_t) = \frac{c_{0t}(1 - n_t) \sigma^2_F}{n_t R + (1 - n_t)(R - c_{1t}) \sigma^2_M}$$ (15)

$$b_1(n_t, c_t) = \frac{n_t R \phi (1 - \beta) \phi + (1 - n_t) c_{2t} \sigma^2_F}{n_t R + (1 - n_t)(R - c_{1t}) \sigma^2_M}$$ (16)

$$b_2(n_t, c_t) = \frac{n_t R \frac{R - \phi}{R - \phi} \beta}{n_t R + (1 - n_t)(R - c_{1t}) \sigma^2_M}$$ (17)
The extent to which the market clearing price reflects the public $d_t$ or the private $d_{t+1}$ depends on the confidence of the fundamental traders in their signal ($\beta$), the beliefs of the market-based traders about the relationship between market observables and future payoffs ($c_{0t}, c_{1t}, c_{2t}$), the trader’s uncertainty in predicting future payoffs ($\sigma_F^2 t, \sigma_M^2 t$), and the proportion of the market employing fundamental vs market-based information ($n_t$). Naturally, also present is the opportunity cost of investing in the risky asset ($R$) and the AR(1) coefficient of the dividend process ($\phi$).

The 2-choice version of the more general $K$ choice Replicator Dynamic (RD) model found in Branch and McGough (2008) results in the transition equation in

$$n_{t+1} = \begin{cases} 
n_t + r(\hat{\pi}^F_t - \hat{\pi}^M_t)(1 - n_t) & \text{for } \hat{\pi}^F_t \geq \hat{\pi}^M_t \\
n_t + r(\hat{\pi}^F_t - \hat{\pi}^M_t)n_t & \text{for } \hat{\pi}^F_t < \hat{\pi}^M_t \end{cases}$$

(18)

with

$$r(x) = \tanh(\delta x/2).$$

(19)

driving the $n_t$ process. The alternative Discrete Choice Dynamic (DCD) model has as a transition function

$$n_t = \frac{1}{2}(1 + \tanh(\rho(\hat{\pi}^F_t - \hat{\pi}^M_t)/2)).$$

(20)

What distinguishes the two population processes is whether the performance differential determines the innovation in $n_t$ or the level. Under the RD process, the more successful strategy attracts adherents from the less successful strategy, consistent with the process described in Grossman and Stiglitz (1980). Under the DCD, employed in Brock and Hommes (1998) and many related papers, $\hat{\pi}^F_t - \hat{\pi}^M_t$ maps directly into $n_t$ with the superior strategy always employed by the majority of the population.

Define an $n_t$-dependent Rational Expectations Equilibrium (henceforth REE($n_t$)), as the solution in which the beliefs of the market-based traders are consistent with the true market process.
In this case, there exists $b_2$ that solves (23), (26), and (27) so that, for $n \in (0, 1]$,

$$
p_t^* = p_t^*(n_t) = b_1^*(n_t)d_t + b_2^*(n_t)d_{t+1} \tag{21}
$$

$$
b_1^*(n_t) = \frac{n_t(1 - \beta)\phi}{(R - \phi)\left(n_t + (1 - n_t)\frac{\sigma_R^2}{\sigma_M^2}\right)} \tag{22}
$$

$$
b_2^*(n_t) = \frac{n_t\beta + (1 - n_t)\frac{\sigma_R^2}{\sigma_M^2}}{(R - \phi)\left(n_t + (1 - n_t)\frac{\sigma_R^2}{\sigma_M^2}\right)} \tag{23}
$$

$$
c_1^*(n_t) = \frac{R}{(R - \phi)b_2^*(n_t)} = \frac{R\left(n_t + (1 - n_t)\frac{\sigma_R^2}{\sigma_M^2}\right)}{n_t\beta + (1 - n_t)\frac{\sigma_R^2}{\sigma_M^2}} \tag{24}
$$

$$
c_2^*(n_t) = \frac{\phi}{R - \phi}(R - c_1^*(n_t)) = -\frac{n_tR(1 - \beta)\phi}{(R - \phi)\left(n_t\beta + (1 - n_t)\frac{\sigma_R^2}{\sigma_M^2}\right)} \tag{25}
$$

$$
\sigma_F^2(n_t)^2 = \left((1 - \beta)^2\left(\frac{R}{R - \phi}\right)^2 + b_2^*(n_t)^2\right)\sigma_{\epsilon}^2 \tag{26}
$$

$$
\sigma_M^2(n_t)^2 = b_2^*(n_t)^2\sigma_{\epsilon}^2. \tag{27}
$$

For $n_t = 0$, $b_1^*(0) = \phi/(R - \phi)$, $b_2^*(0) = 0$ as derived from the consistent solution $c_1^*(0) = 0$ and $c_2^*(0) = R$.

Let $p_t^0$ and $p_t^1$ represent the price at the two information efficiency extremes with, for $n_t \neq 0$, $p_t^0 \equiv p_t^*(1)|_{\beta=0} = \frac{\phi}{R - \phi}d_t$ and $p_t^1 \equiv p_t^*(n_t)|_{\beta=1} = \frac{1}{R - \phi}d_{t+1}$. At the population extreme, $p_t^F \equiv p_t^*(1) = \frac{(1 - \beta)\phi}{R - \phi}d_t + \frac{\beta}{R - \phi}d_{t+1}$. The opening for profitable employment of the market-based information is the fact that $p_t^F \in [p_t^0, p_t^1]$, introducing predictability in the price. The presence of the market-based traders moves the market towards the efficient market price, as reflected in $p_t^*(n_t) \in [p_t^F, p_t^1]$ with $\lim_{n_t \to 0} p_t^*(n_t) = p_t^1$. Since $p_t^*(0) = p_t^0$ there is a Grossman and Stiglitz (1980) type discontinuity at $n_t = 0$.

Observe that $b_1^*(n_t) + \phi b_2^*(n_t) = \phi/(R - \phi)$. Let $x_t = x(n_t) = n_t(1 - \beta)/\left(n_t + (1 - n_t)\frac{\sigma_R^2}{\sigma_M^2}\right)$ then the REE$(n_t)$ price can be expressed as $p_t^* = \frac{1}{R - \phi}(x_t\phi d_t + (1 - x_t)d_{t+1})$ with $x_t \in [0, 1]$. The REE$(n_t)$ price can be interpreted as the present discounted value reflecting the aggregate of the market’s forecast of future dividends. The extent to which the REE$(n_t)$ price reflects the public $d_t$ or the private $d_{t+1}$ depends on the traders. Referring back to (9), the REE$(n_t)$ solution is robust to
certain irrationality or information limitations on the fundamental model. The fundamental model can rationally employ the correct $p^*_t(n_t)$ price function. Alternatively, without consequence on the REE($n_t$) solution, they may mistakenly employ $p^*_t(m_t)$ for $m_t \neq n_t$, or naively employ the $p^F_t$, $p^0_t$, $p^1_t$, or any other (9) consistent price structure without consequence on the market clearing price.

The same latitude cannot be extended to the market-based model. The REE($n_t$) solution depends on the market-based traders correctly employing the REE($n_t$) consistent belief, $c^*(n_t)$. Before turning to the learning process of (3) and (4) by which the coefficients of the market-based model are updated from market realizations, it is appropriate to ascertain whether the traders can deduce $c^*$ analytically from their knowledge of the market. As seen in (25), $c^*_2$ can be expressed in terms of $c^*_1$. For a known zero net supply of the risky asset, the traders can determine that $c^*_0 = 0$. That leaves only $c^*_1$ to be derived by the traders. As seen in (24), solving for $c^*_1$ requires knowledge of $n_t$. Reasonably, $n_t$ is not directly observable and thus excluded from $Z_{it}$. The question becomes whether some $n^{eq}$ value can be identified as part of a REE solution, which can be answered by determining whether a fixed point $n^*$ exists in the dynamic system.

Without knowledge of $n_t$, can $c^*_1(n_t)$ be learned? For $\lambda_t = 1/t$, as would be consistent with the Marcet and Sargent (1989b) least-squares learning, then, according to Proposition 1, $c^*(n_t)$ is a locally stable fixed point of the learning process so that for some fixed $n$, then $c_t \to c^*(n)$. If there is some fixed point $n^*$ to the population process then potentially there exists an attainable REE solution.

**Proposition 1.** For a fixed $n$ and with $\lambda_t = 1/t$, the dynamic system (1) through (5) converges asymptotically to the REE($n$).

**Proof.** See Appendix

Forward looking traders would select which strategy to employ, fundamental or market-based, based on a forecast of the performance of the strategy’s employment in the current period. Define performance in terms of individual profit,

$$\pi^k_{it} = q^k_{it}(p_{t+1} + d_{t+1} - Rp_t).$$ (28)
For $n_t \in (0, 1]$,

\begin{align}
E(\pi_t^F) &= (1 - n_t)\Delta_t \\
E(\pi_t^R) &= -n_t\Delta_t
\end{align}

(29) (30)

so that $E(\pi_t^F - \pi_t^M) = \Delta_t$, where

$$
\Delta_t = \Delta(c_{1t}, n_t) = \left(\frac{n_t(1 - \beta)R + (1 - n_t)(R - c_{1t})\frac{\sigma_F^2}{\sigma_M^2}}{n_tR + (1 - n_t)(R - c_{1t})\frac{\sigma_F^2}{\sigma_M^2}}\right) \left(\frac{R}{R - \phi}\right)^2 \frac{(R - c_{1t})\beta\sigma_F^2}{\sigma_M^2}.
$$

The REE($n_t$) expected profit differential, $E(\pi^*_F - \pi^*_M)$, solves to

$$
\Delta^*(n_t) = -\left(\frac{1 - \beta}{n_t + (1 - n_t)\frac{\sigma_F^2}{\sigma_M^2}}\right)^2 \left(\frac{R}{R - \phi}\right)^2 \frac{n_t\sigma_F^2}{\sigma_M^2}.
$$

(31)

That $\Delta^*(n_t) < 0$ for all $n_t \neq 0$ reveals the benefit to extracting filtered information from the the REE market over direct access to noisy information. The fundamental traders only profit in the presence of error in the market-basted traders’ model, as $c_{1t}$ deviates sufficiently from $c^*_1(n_t)$.

The REE($n_t$) solution is the fixed point to the learning process. A fixed point to the entire dynamic system thus requires the REE($n_t$) solution combined with a fixed point to the population process. The fixed point condition depends on the population regime.

**Proposition 2.** For $\rho \in [0, \infty)$, there exist a unique fixed point $n^f$ to the dynamic system (1) through (7) and (20).

*Proof.* Under the DCD population process, $f(\hat{\pi}_t^F - \hat{\pi}_t^M, n_t) = f(\hat{\pi}_t^F - \hat{\pi}_t^M)$ and at the REE($n_t$), $\hat{\pi}_t^F - \hat{\pi}_t^M = \Delta^*(n_t)$. For $\rho < \infty$, $f(x)$ is continuous and monotonically increasing in $x$. A fixed point is solution $n^f$ such that $n^f = f(\Delta^*(n^f))$. Since $\lim_{n_t \to 0} \Delta^*(n_t) = 0$ and $\Delta^*(n_t)$ is monotonically decreasing as $n_t$ increases to one, a unique $n^f$, $0 < n^f \leq 1/2$, such that $n^f = f(\Delta^*(n^f))$ exists.

Figure 1 captures the existence of the fixed point under the DCD population process. Since the
slope of \( f(\pi^F - \pi^M) \) evaluated at \( \pi^F - \pi^M = 0 \) increases with \( \rho \) in Figure 1, the value of \( n^{fp} \in (0, 1/2] \) decreases with increasing \( \rho \). At the extreme, \( \rho = 0 \) results in a horizontal \( f(\pi^F - \pi^M) \) and \( n^{fp} = 1/2 \) while \( \rho \to \infty \) approaches a step function in \( f(\pi^F - \pi^M) \) so that \( n^{fp} \to 0 \). Long memory in the performance measures ensures stability of the fixed point. If the population is adaptive in its choice of strategy, with \( \mu_t = 1 \), then the fixed point becomes unstable for sufficiently high \( \rho \).

**Proposition 3.** No fixed point exists for dynamic system (1) through (7) and (18).

**Proof.** Under the RD population process, the fixed point condition requires the existence of an \( n^{fp} \) such that \( n^{fp} = f(\Delta^*(n^{fp}), n^{fp}) \), a condition that reduces to simply \( n^{fp} \) such that \( \Delta^*(n^{fp}) = 0 \). Since no such \( n^{fp} \) exists, there can be no fixed point to the RD population process.

The fixed point condition for the population process requires that \( \Delta(c_{1t}, n_t) = 0 \). A fixed point \( n^{fp} \) exists as a function of \( c_{1t} \) so that \( n^{fp} = n^*(c_{1t}) \). The existence of a REE equilibrium, attainable either analytically or through learning depends on the existence of an \( (n^{fp}, c_1) \) combination for which \( n^{fp} = n^*(c_1(n^{fp})) \). Such a point does not exist since, for \( c_{1t} = c^*_1(n_t) \), \( \Delta^*(n_t) < 0 \) for all \( n_t \in (0, 1] \) and \( E(\pi^F_t - \pi^M_t) > 0 \) for \( n_t = 0 \).

The dynamic system is thus characterized by the existence of a REE\( (n_t) \) in beliefs that (a) depends on knowledge of an unobservable \( n_t \) and (b) is the source of instability in \( n_t \). It is thus
reasonable to think of this model as existing perpetually out of equilibrium. This investigation considers the impact on the market of reasonable boundedly rational trader behavior given the non-equilibrium market condition.

3 Literature

The potential information advantage of the market-based traders introduces into the model a source of market tension that is rare in the existing literature. It is common to endow those traders who bring fundamental private information into the market an absolute information advantage, handicapped in generating returns by a higher cost of implementation, (examples, including BS and BH). The fundamental traders of the current model are heterogeneously informed, each trader receiving a signal based on fundamental information but subject to idiosyncratic error.\(^\text{10}\) The resulting in a Hellwig (1980) type filtering of the idiosyncratic errors of the individual fundamental traders reveals, when efficient, the underlying information. The market-based traders have the advantage of market aggregation if they can correctly extract the information present in the price. With the correct model, they are better informed than the fundamental traders. The current model thus captures a different contrast between trading strategies; fundamental information with its proclivity to individual error and market based information and its dependence on exploitable information in the price.

As a dynamic financial market model allowing traders to switch between two strategies, the model invites comparison to Brock and Hommes (1998) (henceforth, BH). As a model in which uninformed traders extract information from those who are privately informed through the price, the model invites comparison to Grossman and Stiglitz (1980) (henceforth, GS). The nature of the risky asset and of the information are reasonable and attractive features for a financial market model which combine to produce financial market characteristics distinct from the reference market models.

In common with both BH and GS are the two groups of traders who choose between fundamental information and information extraction from market data. The risky asset is infinitely lived with its

\(^{10}\)As in Brock and LeBaron (1996), Goldbaum (2003) and Goldbaum (2006).
fundamental value determined by a stochastic dividend process. This type of risky asset is employed by BH, but is absent from the typical dynamic models based on GS. Multi-period GS-based models tend to rely on a single period lived asset with an unknown terminal value. Multi-period lived assets introduce price and return dependence between periods that allow for persistent price deviations to emerge as cannot be accomplished with single-period lived assets.

GS use an uncertain supply to handicap the information extraction from the observed price. The devise is effective because it impacts the price without altering the end of period payoff. With the infinitely lived asset of the current model, investors would need to be concerned with the capital gains attributable to fluctuations in the supply of the asset. The devices is also unnecessary in the current environment where $n_t$ is unknown to the traders.

The fundamental traders of the current model are provided with useful private information. In trading, this information is reflected in the price, thus providing an opportunity for those who have not received the information directly to extract the information from the price. This, of course, is an important feature of GS, but is not an element of the BH model. The BH uninformed traders are limited to employing only past, rather than concurrent, information.\footnote{In the BH model, the fundamental traders trade based on the fundamental value computable from public information. They do not possess or trade on private information. As a result, there is no useful information to extract from the price that cannot be obtained from public information.}

4 Boundedly Rational

The market-based traders, unable to form a rational expectations about the relationship between market observables and payoffs employ least-squares learning. Different levels of rationality can be considered. Traders aware of the market and its structure, as previously observed, can deduced that $c_0 = 0$ regardless of the unobservable $n_t$. Additionally, they may choose to incorporate the feature of the REE($n_t$) and impose $c_{2t} = c^*_2(c_{1t})$ according to (25) so that they only estimate one unknown coefficient, $c_{1t}$. At the other extreme, the traders can let the data drive the estimates of all three regression coefficients.

Traders who are aware of the market structure will be aware that the least-squares learning process is not asymptotically consistent with the setting. There is no fixed point to the dynamic
system to which the learning process can potentially converge. The least-squares learning will perform best for the market-based traders if \( n_t \) is relatively stable over time so that the relationship between price and payoff is approximately unchanged. The aware trader recognizes that the more accurate is the market-based model due to consistency in the data across time, the greater the incentive for a performance induced decline in \( n_t \). A reasonable accommodation may be for the trader to employ constant gains in the coefficient updating algorithm, setting \( \lambda_t = \lambda \) in (3) and (4), thereby placing greater weight on the more recent data.

Traders track the relative performance of the two trading approaches as an indicator of how they will perform in the coming period. They can choose to employ the least-squares consistent gains in (6) with \( \mu_t = 1/t \) thereby giving equal weight to all observations. A reasonable alternative would be to emphasize more recent performance by employing a constant gain parameter, setting \( \mu_t = \mu \).

Another challenge for the traders is how to evaluate their error associated with their forecast of the payoff. The demand equation submitted to the Walrasian auctioneer includes the variance associated with the forecast error. As reflected in (35) and (36), the values can be derived for each forecast strategy, but only with knowledge of the true \( n_t \)-dependent pricing relationship. The boundedly rational alternative is to have the traders estimate the values. Simulation reveals that the choice of how the traders estimate the uncertainty of the two approaches has profound effects on the market’s behavior.

5 Model Dynamics

5.1 A simplified version

Analysis of a reduced version of the dynamic system allows for anticipation and greater understanding of the simulation output to follow. The system includes seven transition equations but with the imposition of some constraints, much of what transpires in the dynamic system can be understood from the phase space of \( n_t \) and \( c_{1t} \). For tractability, assume a high degree of rationality in the market-based model so that \( c_{0t} = 0 \) and \( c_{2t} = c_{2t}^*(c_{1t}) \). Under these conditions, \( c_{1t} \) is the
only parameter in the dynamic learning process of the market-based model. Further, impose that beliefs about the conditional errors are consistent with the REE($\eta_t$) solution so that $\hat{\sigma}^2_{tk} = \sigma_k^*(\eta_t)^2$ for $k \in \{F, M\}$. A simplified population process, with $\mu_t = 0$, further facilitates evaluation.

The $c_{1t}$ process is at a fixed point and a REE($\eta_t$) if $c_{1t} = c_1^*(\eta_t)$. At the REE($\eta_t$), the market-based model correctly reflects the relationship between the observables $d_t$ and the expected payoff of the following period, $E(p_{t+1} + d_{t+1})$. The function $c_1^*(\eta_t)$ is monotonically increasing for $0 < \eta_t \leq 1$ with $c_1^*(\eta) \rightarrow R$ for $\eta \rightarrow 0$ and $c_1^*(1) = R/\beta$.

The population process is at a fixed point if $\Delta(c_{1t}, \eta_t) = 0$. Let $c_1^+ (\eta_t)$ and $c_1^- (\eta_t)$ represent to two functions capturing combinations of $c_{1t}$ and $\eta_t$ consistent with $\Delta(c_{1t}, \eta_t) = 0$. For $0 < \eta_t \leq 1$ the former is a monotonically increasing function and everywhere above $c_1^*(\eta_t)$,

$$c_1^+(\eta_t) = R \left( 1 + (1 - \beta) \frac{\eta_t}{1 - \eta_t} \frac{\sigma^2_M}{\sigma^2_F} \right),$$

while the latter is a constant, located below $c_1^*(\eta_t)$, $c_1^- = R$. Expected profits are zero at $c_{1t} = c_1^+ (\eta_t)$ because the resulting market clearing price is the efficient market price, $p_1^*$, so that expected profits are zero regardless of the individual trader’s position taken in the market. Expected profits are zero at $c_{1t} = c_1^- (\eta_t)$ because the market traders’ expect the risky asset to offer the same as the risk-free bond and thus there is no trading at the market clearing price.

A final relevant function included in the phase space is $\bar{c}_1(\eta_t)$. Let $\bar{c}_1(\eta_t)$ represent the function capturing combinations of $c_{1t}$ and $\eta_t$ such that $\eta_t R + (1 - \eta_t)(R - c_{1t}) \frac{\sigma^2_F}{\sigma^2_M} = 0$,

$$\bar{c}_1(\eta_t) = R \left( 1 + \frac{\eta_t}{1 - \eta_t} \frac{\sigma^2_M}{\sigma^2_F} \right).$$

This expression appears in the denominator of the two pricing coefficients, $b_1(c_{1t}, \eta_t)$ and $b_2(c_{1t}, \eta_t)$. Its negative is the slope of the aggregate demand function so that when it is zero, the aggregate market demand function is horizontal and different from zero, producing an infinite market clearing price (based on a zero net supply). The function $\bar{c}_1(\eta_t)$ is upward sloping and everywhere above $c_1^+(\eta_t)$. As the function is approached from below or from the right the $p_t(c_{1t}, \eta_t) \rightarrow \pm \infty$.

Above $\bar{c}_1(\eta_t)$, the combination of $\eta_t$ and $c_{1t}$ do not allow for a reasonable market clearing price.
The fragility of the market in the vicinity of $\tau_1(n_t)$ is the consequence of the excessive influence of the market-based traders. As a group, they have an upward sloping demand function in price. A increase in the price is interpreted as an indication of good news about the underlying $d_{t+1}$. At $c_{1t} = c_1^*(n_t)$, the market-based model correctly accounts for the presence of the market-based traders in the market and their influence on the price and the aggregate demand for the risky asset remains downward sloping in $p_t$. For $c_{1t} > c_1^*(n_t)$, the market-based model projects too great a deviation in $d_{t+1}$ from the observed $p_t$ and thus takes too large a position. For $c_{1t} > \tau_1(n_t)$, the position produces an upward-sloping demand function.\textsuperscript{12}

Whether the market can be relied on to be reasonably well behaved depends on whether the system can be relied upon to remain far below $c_1^-(n_t)$. The traders themselves cannot be relied upon to recognize dangerous market conditions. Since there exists some $0 < n_1 \leq 1$ for which $c_{1t} = c_1^*(n_1)$, any $c_{1t} \in (R, R/\beta]$ is reasonable and potentially correct. There also exists some $0 < n_2 < n_1$ for which $c_{1t} = \tau_1(n_2)$ so that the same $c_{1t}$ can potentially generate substantial mispricing.

If $c_{1t}$ is updated according to a least-squares learning consistent process, then $c_1^*(n_t)$ is an attractor for $c_{1t}$ given $n_t$. For $c_{1t}$ between $c_1^-(n_t)$ and $c_1^+(n_t)$, $E(\Delta(c_{1t}, n_t)) < 0$ so that $n_t$ tends to decline. In this range, the market-based model, while not exactly right for extracting information from the price, is sufficiently correct so that the market-based forecasts are more accurate than the average fundamental trader relying on a noisy signal. Outside this range, with $c_{1t} < c_1^-$ or $c_1^+(n_t) < c_{1t} < \tau_1(n_t)$, the inaccuracy in the market-based model is sufficiently large that the average fundamental trader expects to earn profits at the expense of the market based traders and $n_t$ tends to increase in this region.

All four functions of the phase space converge towards $R$ as $n_t \to 0$ but because of the discontinuity at $n_t = 0$, none take a value of $R$ at $n_t = 0$. Therefore, though the four functions come arbitrarily close, they never intersect. The failure of $c_1^*(n_t)$ to intersect with either $c_1^-(n_t)$ or $c_1^+(n_t)$ graphically captures the absence of a fixed point to the dynamic system.

\textsuperscript{12}As an alternate interpretation, for $c_{1t} > c_1^*(n_t)$, the market-based model can be seen as underestimating the influence of the market-based traders on the price since, for $c_{1t} < R/\beta$, there exists $n > n_t$ such that $c_{1t} = c_1^*(n)$.  

18
Figure 2: Phase Space in $n_t$ and $c_{1t}$. $c_1^*(n_t)$ is the REE($n_t$) value of $c_{1t}$ and is the attractor to the learning process for a given $n_t$. For $c_1^- < c_{1t} < c_1^+(n_t)$ the market-based model is sufficiently accurate to earn profits at the expense of the fundamental strategy so that $n_t$ declines. For $c_{1t} < c_1^-$ and for $c_1^+(n_t) < c_{1t} < c_1^*(n_t)$ the fundamental strategy dominates the market-based strategy so that from these regions $n_t$ is increasing. Above $c_1(n_t)$, the aggregate demand curve for the risky security is upward sloping and no positive price exists to clear the market. The boundaries of the phase space are affected by how the traders estimate the conditional variance associated with their forecasts. The dashed lines reflect an alternate specification for which $c_{1t} \neq c_1^*(n_t)$ is recognized when calculating the market-based model error.
5.2 Simulations

The current project explores the behavior of the market while relaxing the assumption used to generate the analytical phase space.

Let $p^1_t$, the strong-form efficient market price, be the standard against which the market price is evaluated. Let $|p_t - p^1_t|$ be the measure of market efficiency. In general

$$p_t - p^1_t = (b_{1t} + \phi b_{2t} - \phi/(R - \phi))d_t + (\phi b_{2t} - \phi/(R - \phi))\varepsilon_{t+1}. \tag{34}$$

From (34), there are two sources of deviation from $p^1_t$. At the learning fixed point, $b^*_1(n) + \phi b^*_2(n) = \phi/(R - \phi)$ regardless of $n$ and $\phi b^*_2(n) \to \phi/(R - \phi)$ as $n \to 0$. Thus, the first term of (34) is zero when the market-based model is correct for the given $n_t$, with $p_t = p^*_t$. The second term is the bias in the market price produced by the influence of the demand of the fundamental traders. The second term only converges to zero when $c^*_1 = c^*_1$ and as $n \to 0$ so that $p^*_t \to p^1_t$.

All simulations share the parameter values, $R = 1.02$, $\phi = 0.5$, $\sigma_d = \sigma_\varepsilon = 1$ so that $\beta = 1/2$, and the starting value, $n_0 = 0.75$. Pre-simulation learning on the market-based model takes place on 200 observations generated using a fixed $n_t = n_0$. The simulations explore the impact of altering other aspects of the model, including the parameters $\delta$, $\lambda_t$, and $\mu_t$. Variations will also include features capturing the rationality or sophistication of the market participants in determining how they estimate $c_{2t}$, $\sigma^2_{Mt}$, and $\sigma^2_{Ft}$ as well as comparing the two population processes, RD vs DCD.

Figures 3a through 8 display the evolution of endogenous parameters produced by the simulations. To aid direct comparison, each figure is based on the same underlying randomly generated $\{d_t\}$ series. Each frame plots the time progression of endogenous parameters of the model. Across the top row are plotted the price values $b_{1t} + \phi b_{2t}$ and $\phi b_{2t}$, reflecting the two coefficients in (34). The time series are plotted in green. Unless designated otherwise, the figures cover a time range well into the simulation, demonstrating the asymptotic properties of the market under the particular parameters.

The second row of frames contains the regression coefficients from the market traders’ model, $c_{1t}$ and $c_{2t}$ plotted in green. In all simulations, the value of $c^*_0$ is presumed to be known by the
traders and thus \( c_{0t} = 0 \). When useful, the \( c_{1t} \) and \( c_{2t} \) frames may included plots of \( c_{1t}^*(n_t) \) (in red) and a black line at \( c_{1}^- = R \) and \( c_{2}^- = c_{2}^*(c_{1}^-) = 0 \) which are also the values associated with \( \lim_{n \to 0} c_{1t}^*(n) \) and \( \lim_{n \to 0} c_{2t}^*(n) \), respectively. Also included with \( c_{1t} \) are \( c_{1t}^+(n_t) \) (in cyan) or \( \bar{c}_{1t}(n_t) \) (in blue) based on \( \sigma_{Mt}^2 = \sigma_{Ft}^2 \).

The bottom row presents the population parameter \( n_t \) and the price deviation from efficiency, \( p_t - p_{1t}^1 \).

### 5.2.1 Discrete Choice Dynamics

A fixed point exists to the population process, and thus to the entire dynamic system, under the DCD process. The stability of the fixed point is assured if the traders employ \( \mu_t = 1/t \) in their performance updating, otherwise the fixed point is stable only if \( \rho \) is sufficiently low. Figure 3a shows the early convergence of the system towards the fixed point values of the respective parameter, which is indicated with a horizontal black line. Figure 3b shows the asymptotic properties of the convergence.

Shorten the memory associated with performance and the random component of realized returns becomes, for \( \rho \neq 0 \), the source of noise in \( n_t \) around \( n^{fp} \). Sufficiently large \( \rho \) produce \( n_t \) sufficiently low as to approach or enter the invalid price region by widening the distribution of \( n_t \). In this case, the market is unable to settle into a time independent asymptotic environment around \( c_{1t} = c_{1t}^*(n^{fp}) \). At the default parameters, this occurs at \( \rho = 0.5 \).

### 5.2.2 Replicator Dynamics

The dynamic system is without a fixed point under the RD process. Instead, there is a point of attraction when the system is well behaved with \( n_t \to 0 \) and the remaining parameters consistent with the \( \text{REE}(n_t) \) solution. The \( 
\text{REE}(n_t) \) solution has \( b_{1t} + \phi b_{2t} = \phi/(R - \phi) \) and \( b_{2t} \to \phi/(R - \phi) \) as \( n_t \to 0 \). Both the \( b_{1t} + \phi b_{2t} \) and \( b_{2t} \) frames in Figures Y-Z include a solid black line at \( \phi/(R - \phi) \).

The first RD simulation employs parameters consistent with convergence towards the attractor. The users of the market-based model are highly rational, imposing the condition \( c_{2t} = c_{2t}^*(c_{1t}) \) so that the market-based traders only have to estimate \( c_{1t} \) using (3) and (4). The enforced consistency
Figure 3: Baseline DCD: Convergence to a REE fixed point at $n^{fp} = 0.357$
between $c_{1t}$ and $c_{2t}$ eliminates one source of pricing error by ensuring $b_{1t} + \phi b_{2t} = \phi / (R - \phi)$. This, in turn, ensures that the fundamental forecasts of $p_{t+1} + d_{t+1}$ err only as a consequence of error in their private signal and not due to misperception of the pricing function. The gain parameters in (3) and (6) are least-squares learning consistent with $\lambda_t = \mu_t = 1/t$. The traders’ estimates of $\sigma^2_{Ft}$ and $\sigma^2_{Mt}$ are derived from experience and updated according to (5).

Figure 4 is typical of the time series generated by this environment. The main features are that there is convergence in $n_t$ towards zero and progressive updating of $c_{1t}$ so that it remains close to $c^*_1(n_t)$. As a consequence, $c_{1t}$ converges towards $\lim_{n \to 0} c^*_1(n) = R$. Contributing to the smooth process of convergence is the slow evolution in $n_t$, a product of a relatively small $\delta$. Increase $\delta$ and $n_t$ oscillates while maintaining a process of convergence towards zero.
Figure 5: RD with $\mu_t = 1$

Figure 5 substitutes the long memory of $\mu_t = 1/t$ with the same adaptive beliefs in performance that created the instability of the DCD process, $\mu_t = 1$. The parameters used in Simulation 4 are consistent with the development of the phase space in Figure 2. In this scenario, the convergence of $n_t$ towards zero is halted with $n_t$ hovering around 0.4. The explanation originates with the inability to completely predict $p_{t+1}$ with time $t$ available information since its realization depends on $d_{t+2}$. As a result, realized profits continue to deviate from expectation even for $c_{1t} = c_1^*(n_t)$. With a short memory, these realizations generate movement in $n_t$ from period to period that undermine the learning of $c_{1t}$ and ultimately the convergence of $n_t$.

Before exploring the implications of constant gains updating in the market-based trader model in the entire dynamic system, it is useful to examine the properties of constant gain learning with a
fixed $n$. Under the regularity conditions of Evans and Honkapohja (2001), the estimated parameters should converge to a fixed unbiased distribution around the true parameter. As seen in Figure this generally does not prove true. The source of the skewed distribution is the asymmetry in the price impact generated by deviations from $c^*_1(n)$ with $c_{1t} > c^*_1(n)$ producing a bigger price deviation than $c_{1t} < c^*_1(n)$ by the same magnitude.

Switching to constant gains in the updating of the market-based model parameters with $\lambda_t = 0.01$ generates a different kind of non-convergence in $n_t$. As seen in Figure 6, after a period of learning, $c_{1t}$ settles into a stable the distribution around $c^*_1(n_t)$, moving over time to track $c^*_1(n_t)$. The narrow distribution in $c_{1t}$ tends to favor the market-based model. The constant gain becomes a liability because without improvement in the estimate of $c_{1t}$ over time, then as $n_t$ converges towards zero, the distribution in $c_{1t}$ is at some point too wide to remain between $c^-_1$ and $c^+_1$. The resulting mispricing substantially rewards the fundamental model, reversing the progress in $n_t$ with $\hat{\pi}^F - \hat{\pi}^M$ remaining positive for some time while the small profits earned by the market-based strategy accumulate.

Relax the rationality of the trader by decoupling $c_{2t}$ from $c^*_2(c_{1t})$ so that the market-based strategy estimates both coefficients through the learning process of (3) and (4). The consequence is two fold. The possible inconsistency between $c_{1t}$ and $c_{2t}$ introduced a new source of error in the market-based model. There is additionally a price impact beyond simply adding to the magnitude of the error. As depicted in the first frame of Figure 7, the constraint $b_{1t} + \phi b_{2t} = \phi/(R - \phi)$ need no longer hold. Under this environment, the market clearing pricing function is no longer consistent with the pricing function of the fundamental model, introducing a new source of error. The consequence of this is seen in (13) and (14) where the coefficient on $d_t$ is no longer equal to zero. The market does not properly price even the observable component of price. Relative to the base simulation, price deviations from the efficient price show considerably more volatility clustering with greater volatility coinciding with deviations from $b_{1t} + \phi b_{2t} = \phi/(R - \phi)$.

A more sophisticated fundamental trader accounts for the market-based traders distortion to the price, as though $b_{1t}$ and $b_{2t}$ were know and accounted for. This sophisticated fundamental trader behavior introduces multiple market clearing prices for sufficiently low values of $n_t$, as depicted
Figure 6: RD with $\lambda_t = 0.01$
Figure 7: RD with $c_{2t}$ free from $c_{1t}^2$.
Figure 8: Roots of $b_2$ in the presence of rational fundamental traders who account for the deviation from $b_{1t} + \phi b_{2t} = \phi/(R - \phi)$ in Figure 8. Rather than imposing discipline in the pricing of the asset, the pricing error are substantially larger than with the less sophisticated fundamental traders.

The traders estimate the conditional variance of the pricing error associated with their chosen model. The ratio of the estimated variance, $\hat{\sigma}^2_Mt/\hat{\sigma}^2_Ft$, affect the coefficients of the pricing equation, $b_{1t}$ and $b_{2t}$. The baseline updating mechanism of (5) has the traders updating the estimated variance based on observation. In a well behaved market such as that produced by (Sim 3), substituting the REE($n_t$) variance values, $\sigma^2_Ft = \sigma^2_F(n_t)^2$ and $\sigma^2_Mt = \sigma^2_M(n_t)^2$ of (26) and (27) has little impact since the estimates track closely to the REE($n_t$) values. The ratio $\sigma^2_F(n)^2/\sigma^2_M(n)^2$ is monotonically increasing in $n$. At the default simulation parameter values, the ratio ranges from 1.52 to 3.08.

The impact on the behavior and convergence of the market can be seen in (32) and (33) where the inverse ratio appears in the formulas for $c^*_{1t}$ and $\bar{c}_{1t}$. Increasing relatively uncertain among the
employers of the market-based model decreases their price impact (because they take a smaller position) and increases the regions between $c_1^+(n_t)$ and $c_1(n_t)$ and between $c_1(n_t)$ and $\bar{c}_1(n_t)$, facilitating convergence towards the attractor. Highly confident market-based traders take larger positions which generate larger price deviation for the same $c_1$ estimate.

Allowing for deviations from the REE($n_t$) solution, for $c_{2t} = c_2^*(c_{1t})$, the conditional variances become

$$\sigma^2_F(n_t, c_{1t}) = \left( (1 - \beta) \left( \frac{R}{R - \phi} \right)^2 + b_2^2(n_t, c_{1t}) \right) \sigma^2_\epsilon \tag{35}$$

$$\sigma^2_M(n_t, c_{1t}) = \left( \left( \frac{R}{R - \phi} - c_{1t} b_2(n_t, c_{1t}) \right)^2 + b_2^2(n_t, c_{1t}) \right) \sigma^2_\epsilon \tag{36}$$

Incorporating this into the model introduces the error in $c_{1t}$ into the time $t$ market-based trader uncertainty, which contributes to the stability of the market by decreasing the size of the position in the market. The greater stability is suggested by the higher values of $c_1^+(n_t)$ and $\bar{c}_1(n_t)$, captured with the dashed lines designated $c_1^+'$ and $\bar{c}_1'$ respectively in Figure 2. There is a value $n_\prime$ such that for $n_t > n_\prime$ the invalid price region does not exist. Thus, regardless of how large is $c_{1t} > R$, for there exists a positive market clearing price and for $n_t > n^+$, the market-based strategy is profitable. Similarly, there is a value of $n^{+\prime}$ such that for $n_t > n^{+\prime}$ the market-based strategy is always profitable.

Employment of a constant gains parameter in (5) might make sense to the trader to accommodate changing conditions. This may seem most appropriate in a setting where $n_t$, and thus $\sigma_k^2(n_t)$, change quickly over time.
References


Chiarella, C., and X. He (2001): “Asset price and wealth dynamics under heterogeneous expectations,”.


Part I

Proof of Proposition 1

*Proof*. Under the regularity conditions (see Marcet and Sargent (1989b); p342-343), the stability of the learning process can be established from the stability of $T(c) - c$ where $T(c)$ maps $c$ into the projection coefficients. From (24) and (25),

$$c_1 = \frac{R}{R - \phi} \frac{1}{b_2}$$

and

$$c_2 = \frac{\phi}{R - \phi} (R - c_1)$$

so that, according to (17),

$$T(c_1) = \frac{nR + (1 - n)(R - c_1) \sigma^2}{\sigma_{\tilde{M}}^2} \frac{1}{n\beta}$$
\[ T(c_2) = -\frac{\phi}{R - \phi} \left( \frac{nR(1 - \beta) + (1 - n)(R - c_1)\frac{\sigma_f^2}{\sigma_M^2}}{n\beta} \right) \]

The eigenvalues of the Jacobian, \( \frac{\partial[T(c) - c]}{\partial c} \), are \( \{-1, -1 - \frac{1-n}{n}\frac{\sigma_f^2}{\sigma_M^2}\} \), which are both less than zero. The learning process is thus locally stable so that \( \Pr(\left| c_t - c^* \right| > \delta) \xrightarrow{\delta} 0 \) for \( \delta > 0 \).